

Matching branches of non-perturbative conformal block at its singularity divisor

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Conformal block is a function of many variables, usually represented as a formal series, with coefficients which are certain matrix elements in the chiral (e.g. Virasoro) algebra. Non-perturbative conformal block is a multi-valued function, defined globally over the space of dimensions, with many branches and, perhaps, additional free parameters, not seen at the perturbative level. We discuss additional complications of non-perturbative description, caused by the fact that all the best studied examples of conformal blocks lie at the singularity locus in the moduli space (at divisors of the coefficients or, simply, at zeroes of the Kac determinant). A typical example is the Ashkin-Teller point, where at least two naive non-perturbative expressions are provided by elliptic Dotsenko-Fateev integral and by the celebrated Zamolodchikov formula in terms of theta-constants, and they are different. The situation is somewhat similar at the Ising and other minimal model points.

1 Introduction

Conformal blocks are the central objects in $2d$ conformal theories [1]: they are holomorphic constituents of the correlation functions, the latter being decomposed into bilinear combinations of the conformal blocks with different internal (intermediate) dimensions. Another ingredient of the theory are the structure constants, defining the coefficients in these expansions (for many purposes it is convenient not to include them into the normalization of conformal blocks, which is then chosen in some other way, more suitable from the point of view of complex analysis). The correlation function can be decomposed in several different ways, and the corresponding conformal blocks are related by *linear* transformations called modular transforms. Through the free fermion formalism [2], certain generating functions are interpreted as τ -functions of the conventional integrable systems and the hierarchies of KP/Toda type, generalization of this formalism to the WZNW model [3, 4] should provide a description as non-Abelian τ -functions of [5]. Long ago the conformal blocks were interpreted as *states* in the Hilbert space of $3d$ Chern-Simons theory [6], thus providing an important ingredient of modern QFT approaches [7, 8, 9, 10, 11] to knot theory [12]. More recently, the AGT relations [13, 14, 15, 16, 17, 18, 19] provided yet another interpretation of conformal blocks and their straightforward q -deformations [20, 21, 22] as sums over instantons in respectively $4d$ and $5d$ Yang-Mills theories with extended supersymmetry [23, 24, 25]. The AGT correspondence proved to be useful in both direction: say, for using the sums over instantons for analysis of minimal models [15, 26, 27] and for using the Zamolodchikov solution [28] for analysis of instanton contributions in Seiberg-Witten theory in the conformal point [29, 30]. All these applications to quantum field theories in various dimensions explain the central role of conformal blocks in theoretical physics and the need for their thorough investigation. It was moved far enough, but, unfortunately, unfinished by Al.Zamolodchikov [31, 28, 32].

In this paper we concentrate on, perhaps, the simplest non-trivial of all conformal blocks: the 4-point spherical one, usually defined as a formal series in the double ratio $x = \frac{(x_2-x_1)(x_3-x_4)}{(x_3-x_1)(x_2-x_4)}$ of the four points on the Riemann sphere,

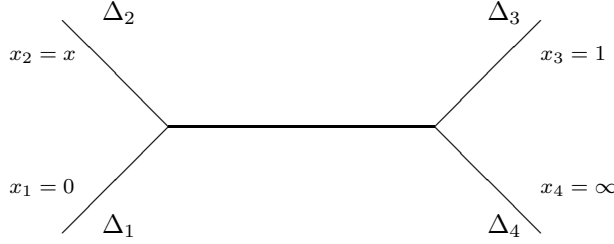
$$B(x) = \sum_{k=0}^{\infty} B_k x^k \quad (1.1)$$

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The coefficients B_k depend on the four external dimensions $\Delta_1, \dots, \Delta_4$, on one internal dimension Δ and on the central charge c .



The series is not everywhere convergent, thus there is a question of what is the function $\mathcal{B}(x)$, which has $B(x)$ as its formal expansion near $x = 0$. We call this function *non-perturbative conformal block* and give a short summary of its already known properties.

As a function of x , the 4-point spherical conformal block is believed to be analytic function ramified just at three points: 0, 1, ∞ , with no essential singularities. Moreover, in the *rational* conformal models ramifications are of finite orders, thus such conformal blocks are actually the Belyi functions [33], appearing in consideration of Grothendieck's *dessins d'enfant* and the equilateral triangulations. The ramified coverings of CP^1 , defined by the Belyi functions, are arithmetic curves, and their description in terms of conformal models and rational dimensions is a newly emerging interesting problem related to description of the universal moduli space [34, 35] and to modern theory of the Hurwitz numbers [36] and the Hurwitz τ -functions [37, 38].

Coming back to non-perturbative $\mathcal{B}(x)$, the convergence of series (1.1) in x is not uniform in the other parameters (Δ_i, Δ, c) , and this makes the entire function of all these variables quite sophisticated. $\mathcal{B}(x)$ is definitely non-trivial: it has many branches and changes under modular transformations. Moreover, this change can be represented as an integral transform in the internal dimension Δ :

$$\begin{aligned} \mathcal{B}_\Delta(1-x) &= \sum_{\Delta'} \mathcal{M}_\Delta^{\Delta'} \mathcal{B}_{\Delta'}(x) \\ \mathcal{B}_\Delta\left(\frac{x}{x-1}\right) &= \sum_{\Delta'} \mathcal{N}_\Delta^{\Delta'} \mathcal{B}_{\Delta'}(x) \end{aligned} \tag{1.2}$$

and in fact the non-perturbative modular kernels $\mathcal{M}_\Delta^{\Delta'}$ and $\mathcal{N}_\Delta^{\Delta'}$ are studied considerably better than $\mathcal{B}_\Delta(x)$ itself. The very fact that an x -independent modular kernel exists is highly non-trivial. This happens only if the intermediate (internal) dimensions Δ is chosen as a parameter in integral transformation (nothing like this would happen if we tried to use, say, one of Δ_i or c), and it reflects the associativity property of the operator product expansion of conformal field theory (it is also referred to as duality), or of the co-product in the Virasoro algebra, the modular kernel being the counterpart of the Racah coefficients (or $6j$ -symbols) in the theory of finite-dimensional Lie algebras.

In the context of quantum field theory and, in particular, in CFT one usually considers a given set of fields, i.e. fixes the set of external dimensions Δ_i , while Δ remains arbitrary: it is common in this context to study the dependence on Δ , but not so common to pay equal attention at dependencies on Δ_i or c , i.e. at the point in the "space of theories". Still, in modern stringy approaches it is more than natural just to discuss the $\mathcal{B}(\mu|x)$ on entire $6_C d$ -space \mathcal{M} of parameters $\mu = \{\Delta_i, \Delta, c\}$, without distinguishing the 1-dimensional "physical slice" $\mathcal{P} \subset \mathcal{M}$, where $\Delta_i, c = \text{const}$. In fact, the conformal block should be analytically continued not only in x , but also in all these six extra parameters μ . In particular, the modular kernel, which lives on the physical slice and thus already attracted certain attention, is believed to possess non-perturbative corrections as a function of Δ and Δ' [39].

Thus, our goal in this paper is to attract attention at the non-perturbative conformal block $\mathcal{B}(\mu|x)$ not only on the physical slice $\mathcal{P} \subset \mathcal{M}$, but everywhere else, with the purpose of identifying the nature and essence of this important class of special functions. It turns out that even at the first step in this direction one runs into interesting details and this should stimulate more attention to such kind of problems.

There are special choices of external dimensions and central charge, when much seems to be known about $\mathcal{B}(x)$, and it is natural to begin from this point.

(A) For degenerate Verma modules there are null-vector constraints, which imply differential equations for the conformal block as functions of x [1]. The order of the equation is defined by the level of the null-vector, and in these cases, it is, first, finite, and, second, greater than one. The second property means that $\mathcal{B}(x)$ has

different branches and thus actually lives on a non-trivial Riemann surface in the x -space. The first property means that there is only a *finite*-dimensional family of solutions, while one could think that only one of the six parameters μ (that is, a particular external dimension, say, Δ_2) is fixed by the null-vector condition, and a function of five remaining parameters should be a solution.

(B) The conformal block possesses the Dotsenko-Fateev representation in terms of multiple Selberg (generalized hypergeometric) integrals, analytically continued in the numbers N_1, N_2 of integration [17]. However, at N_1 and N_2 fixed they are just integrals and thus can be investigated as non-perturbative quantities by usual means of complex analysis. This is an obvious possibility, and it was not closely looked at because the "integral slice" \mathcal{I} defined by $N_1, N_2 = \text{const}$ is in a sense transversal to the physical slice \mathcal{P} , and one never looked carefully at the conformal blocks in transversal directions.

(C) Finally, there is the celebrated Zamolodchikov duality applicable to a very special (Ashkin-Teller) model with $c = 1$ and $\Delta_i = \frac{1}{16}$. The point is that the operators of this dimension create square-root singularities for free fermions (there are two in the $c = 1$ theory), and thus the correlator of fields $V_{1/16}$ at some points can be identified with the partition function of free fermions (Ising model) on a ramified covering of the original space-time. In the special case of 4-point correlator we get a torus with x -dependent modular parameter τ , and in result the fantastic formula for the conformal block at Zamolodchikov's slice $\mathcal{Z} \subset \mathcal{M}$, with $\Delta_i = \frac{1}{16}$, $c = 1$:

$$\mathcal{B}_\Delta(x) = \text{Tr}_{\text{free fermions}} e^{i\pi\tau L_0} = \frac{q^\Delta}{\theta_{00}(q)} \quad (1.3)$$

where

$$q = e^{i\pi\tau}, \quad x = \frac{\theta_{10}^4(q)}{\theta_{00}^4(q)} \quad (1.4)$$

and $\theta_{00}(q) = \sum_{n=-\infty}^{+\infty} q^{n^2} = \prod_{m=1}^{\infty} (1 - q^{2m})(1 + q^{2m-1})^2$. This is again a well-defined non-perturbative formula, and it is so famous because it is on the physical slice.

In this paper we discuss these examples and interplay between them. Our main observation is that both the minimal models (the typical example, where *all* external states are null-vectors) and Zamolodchikov's model at distinguished value of $\Delta = \frac{1}{4}$ appear to lie at the singular locus $\mathcal{L} \subset \mathcal{M}$ where coefficients of conformal blocks are ratios of two vanishing functions – thus they are ambiguously defined. This is a trivial, but previously underestimated phenomenon, because it is not seen at the physical slice: \mathcal{L} and \mathcal{P} are transversal when intersect. It is quite interesting to see what happens at these intersections. Moreover, it is plausible that future investigations of these singularities will help to understand the non-perturbative dependence of conformal blocks on μ , which is important in order to shed some light on the somewhat mysterious non-perturbative expressions for the modular kernels (postulated by the identification [41, 40] of the Virasoro Racah matrices with those for peculiar representations of $SL(2)$).

In order to understand this phenomenon, one has to study the conformal block as a formal power series in x in the vicinities of the singularities. They occur at the Kac zeroes, i.e. whenever the special relations between the internal dimension Δ and the central charge c are satisfied. More concretely, if one parameterizes the conformal dimensions with the Dotsenko-Fateev parametrization:

$$\Delta = \alpha \left(\alpha - b + \frac{1}{b} \right), \quad c = 1 - 6 \left(b - \frac{1}{b} \right)^2 \quad (1.5)$$

the Kac zeroes in B_k happen at all integer $|m| > 0$, $|n| > 0$ such that $mn \leq k$ at the points

$$\alpha = \frac{1}{2} \left(\frac{n-1}{b} - (m-1)b \right) \quad (1.6)$$

In order to have a non-singular answer, one has to require that the conformal block has additional zeroes at these singularities. Since the conformal block is a function of x , one has to require this for each singular B_k . In fact, as we shall see the Kac zeroes have an embedded structure: once B_k is singular at some k , so are all $B_{i>k}$. Surprisingly enough, once one imposes the condition of vanishing the numerator of B_k at the Kac zero, they also vanish for all $B_{i>k}$! Moreover, despite the answer depends on the direction of approaching the singularity, it is parameterized by just one arbitrary constant.

However, in the rational theories a new phenomenon occurs: there accidentally emerge some higher order poles (higher order Kac zeroes) due to the rational value of the central charge. Again, when this, say, double pole emerges at some k , it is also present at all $B_{i>k}$ until it becomes the third order pole etc. However, this is generically no longer the case for the corresponding zeroes in the numerator which are needed to cancel these

poles: they are typically simple zeroes even in rational conformal theories, i.e. one meets in these cases the actual singularity. It means that in a consistent theory the corresponding structure constant should vanish. This makes the structure of the rational conformal theory in the vicinity of singularity quite sophisticated.

A notable exception is the minimal models. In the case of the minimal models the answer does not depend on the direction of approaching the singularity! This distinguishes these models and makes them unambiguously defined.

In the paper we illustrate the described picture with the concrete examples of the Ashkin-Teller model and the minimal models. Note that the Ashkin-Teller model is the only example manifestly known so far, when the answer is ambiguous near the singularity: the other known examples are the minimal models, when this ambiguity is absent.

2 Series and "non-perturbative effects"

We start with a general discussion of what is the non-perturbative conformal block, i.e. what is the conformal block outside its divergency radius¹. The archetypical example of asymptotic perturbative series

$$F(x) = \sum_{n=0}^{\infty} n! x^n \quad (2.2)$$

The factorial growth here is usually connected to the fact that the perturbation is performed with irrelevant operators, and a non-perturbative answer should have a different asymptotics at large values of fields.

The most obvious way to handle the series (2.2) is to interpret the factorials as the Γ -function and use its integral representation to define the non-perturbative function:

$$\mathcal{F}(x) = \sum_{n=0}^{\infty} \Gamma(n+1) x^n = \int_0^{\infty} \left(\sum_{n=0}^{\infty} (zx)^n \right) e^{-z} dz = \int_0^{\infty} \frac{e^{-z} dz}{1 - zx} \quad (2.3)$$

This answer, however, depends on the choice of the contour connecting 0 and ∞ , and the ambiguity is just a residue at the singularity of the integrand:

$$\mathcal{F}(x) \quad \text{is defined modulo} \quad \oint_{z=x^{-1}} \frac{e^{-z} dz}{1 - zx} \sim \frac{e^{-1/x}}{x} \quad (2.4)$$

An alternative way to define the non-perturbative function $\mathcal{F}(x)$ is through writing an equation satisfied by the perturbative series. In the case of (2.2) the simplest one is the first order differential equation:

$$x \frac{d}{dx} x F(x) = \sum_{n=0}^{\infty} (n+1)! x^{n+1} = F(x) - 1 \quad (2.5)$$

Solutions to this equation depend on one free constant: the freedom is to add (with an arbitrary coefficient) a solution to the homogeneous equation

$$x^2 \frac{dF_0}{dx} = (1-x)F_0(x) \implies F_0(x) = \frac{e^{-1/x}}{x} \quad (2.6)$$

i.e. $\mathcal{F}(x)$ is once again defined modulo $x^{-1}e^{-1/x}$. The slight difference is that in the first approach the coefficient seems to be arbitrary integer, while in the second approach, i.e. when the non-perturbative function is defined as a D -module, as a solution to some linear equation, it can naturally be arbitrary complex number.

Of course, in both cases we considered a *minimal* definition: the non-perturbative function can be lifted to bigger moduli spaces, for example, by considering equations of higher order, which have more solutions. Still,

¹Note here that the divergency radius of the conformal block depends on the chosen variable. For instance, in variable x this radius does not exceed 1, since there is a singularity at $x = 1$. At the same time, if one uses the variable $q = e^{\pi i \tau}$ related to x by the formula

$$x = \frac{\theta_{10}^4}{\theta_{00}^4} \quad (2.1)$$

$$\theta_{10}(q) = \theta_{10}(0|\tau) = 2q^{1/4} \sum_{n=0}^{\infty} q^{n(n+1)} = 2q^{1/4} \prod_{m=1}^{\infty} (1 - q^{2m})(1 + q^{2m})^2$$

the series converge in the entire region where $\mathcal{B}(x)$ is analytic, as was demonstrated by Al.Zamolodchikov [28].

usually there is no a *natural* way to get rid of extra parameters; or, whenever this is done, there should be some *additional* reason for such a restriction. Thus, the moral at this stage is:

$$\text{the non - perturbative function usually depends on extra } \textit{hidden} \text{ parameters} \quad (2.7)$$

We emphasize that this fact is implicit in *any* non-perturbative consideration. It is enough to remind the celebrated example of instantons in Yang-Mills theories: not only they come with the typical non-perturbative weight e^{-1/g^2} , they bring in an additional parameter, not seen at the perturbative level: the θ -angle. In fact, restriction to real angles (i.e. to unimodular *free* coefficients in front of the instanton contribution) is somewhat similar to restriction to integer-valued coefficients in the above example, i.e. is motivated more by the concrete method of analytical continuation than by the essence of the problem. What is important in this example, it emphasizes the physical relevance of additional (perturbatively hidden) parameters: they affect the non-perturbative renormalization [42], whose significance is nowadays well appreciated and widely investigated with the help of Seiberg-Witten theory [43]-[45].

In application to conformal blocks, this *would* imply that $\mathcal{B}(\mu|x)$ is actually $\mathcal{B}(\mu|x|C)$, i.e. the non-perturbative conformal block depends on additional parameters C not seen at perturbative level, and not present in the original expansion (1.1) derived from representation theory of the Virasoro algebra.

However, one should expect that the conformal theory is essentially free [46, 47, 48, 4], therefore there is no room for a factorial growth of expansions in the coupling constants of the *irrelevant* operators. Thus, non-perturbative hidden parameters do not appear in this context, and all ambiguities are of a different nature. As we shall see, this makes the non-perturbative conformal blocks easier comprehensible than non-perturbative effects in more general interacting models of quantum field theory. As already mentioned, the expansion series (1.1) in x are believed to have finite radii of convergence, and in this sense they are somewhat different from (2.2).

In fact, even for (2.2) it is recently shown [38]) that it can be naturally associated with a certain KP/Toda τ -function (arising in a simplified counting problem of the Belyi functions [49]). Since the τ -functions satisfy *quadratic* Hirota relations [50, 2, 5] they are not preserved by *linear* transformations, and this can provide a new interesting tool to restrict non-perturbative ambiguities (perhaps, even to distinguish the choice of $\theta = 0$ in instanton calculus, thus leading to a new kind of θ -problem solution).

3 Perturbative expansion of conformal block

We begin from reminding the basic facts about the perturbative conformal block (1.1).

3.1 Definition of perturbative block for *four* primaries

The basic definition of the 4-point conformal block comes from the bilinear expansion of the correlation function of four primary fields [1]

$$\begin{aligned} & \left\langle V_{\Delta_1, \bar{\Delta}_1}(x_1, \bar{x}_1) V_{\Delta_2, \bar{\Delta}_2}(x_2, \bar{x}_2) V_{\Delta_3, \bar{\Delta}_3}(x_3, \bar{x}_3) V_{\Delta_4, \bar{\Delta}_4}(x_4, \bar{x}_4) \right\rangle = \\ & = \sum_{\Delta, \bar{\Delta}} C_{12}^{\Delta, \bar{\Delta}} C_{34}^{\Delta, \bar{\Delta}} \times G_{\Delta}(\Delta_1, \Delta_2, \Delta_3, \Delta_4; c; x_1, x_2, x_3, x_4) \times \bar{G}_{\bar{\Delta}}(\bar{\Delta}_1, \bar{\Delta}_2, \bar{\Delta}_3, \bar{\Delta}_4; c; \bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4) \end{aligned} \quad (3.1)$$

where $C_{12(34)}^{\Delta, \bar{\Delta}}$ are the structure constants of the theory which are sometimes included in the definition of the conformal block. However, in principle, the structure constants is a separate object, it defines the concrete conformal theory, and it is a separate problem to list all admissible structure constants (they should satisfy additional complicated restrictions like duality). At the same time, the conformal block $G_{\Delta}(\Delta_1, \Delta_2, \Delta_3, \Delta_4; c; x_1, x_2, x_3, x_4)$ is a universal function of four points x_i that depends on 6 parameters: 5 dimensions and the central charge, it encodes only properties of the Virasoro algebra. In fact, it is a non-trivial function of the double-ratio $x = \frac{(x_2 - x_1)(x_3 - x_4)}{(x_3 - x_1)(x_2 - x_4)}$ only:

$$G_{\Delta}(\Delta_1, \Delta_2, \Delta_3, \Delta_4; c; x_1, x_2, x_3, x_4) = \left(\prod_{i < j} x_{ij}^{\zeta_{ij}} \right) B_{\Delta}(x) = \left(\prod_{i < j} x_{ij}^{\zeta_{ij}} \right) \sum_k x^k B_k(\Delta_1, \dots, \Delta_4 | \Delta, c) \quad (3.2)$$

Here $x_{ij} \equiv x_i - x_j$, $\zeta_{12(13)} = 0$, $\zeta_{14} = -2\Delta_1$, $\zeta_{23} = \Delta_4\Delta_1 - \Delta_2 - \Delta_3$, $\zeta_{24} = \Delta_1 + \Delta_3 - \Delta_2 - \Delta_4$, $\zeta_{34} = \Delta_1 + \Delta_2 - \Delta_3 - \Delta_4$. Note that permutations of points gives rise to the modular transformations (1.2). For

instance, upon permuting x_1 and x_3 we obtain

$$B(\Delta_3, \Delta_2, \Delta_1, \Delta_4 | \Delta, c; 1-x) = \sum_{\Delta'} \mathcal{M}_{\Delta}^{\Delta'} B(\Delta_1, \Delta_2, \Delta_3, \Delta_4 | \Delta', c; x) \quad (3.3)$$

From now on we omit the parameters $\Delta_1, \Delta_2, \Delta_3, \Delta_4, c$ in notation of the conformal block unless it may lead to a misunderstanding.

The conformal block expansion (3.1) is derived by using the operator product expansion (OPE)

$$\begin{aligned} V_{\Delta_1 \Delta_2}(x_1, \bar{x}_1) V_{\Delta_2, \bar{\Delta}_2}(x_2, \bar{x}_2) &= \sum_{\Delta} C_{12}^{\Delta, \bar{\Delta}} (x_1 - x_2)^{\Delta - \Delta_1 - \Delta_2} (\bar{x}_1 - \bar{x}_2)^{\bar{\Delta} - \bar{\Delta}_1 - \bar{\Delta}_2} \times \\ &\times \left(\sum_{Y, \bar{Y}} (x_1 - x_2)^{|Y|} (\bar{x}_1 - \bar{x}_2)^{|\bar{Y}|} \beta_Y(\Delta | \Delta_1, \Delta_2) \bar{\beta}_{\bar{Y}}(\bar{\Delta} | \bar{\Delta}_1, \bar{\Delta}_2) \hat{L}_{-Y} \hat{\bar{L}}_{-\bar{Y}} V_{\Delta_2, \bar{\Delta}_2}(x_2, \bar{x}_2) \right) \end{aligned} \quad (3.4)$$

where the bracket at the r.h.s. is a sum over descendants, which are labeled by Young diagrams (Y, \bar{Y}) of all sizes $(|Y|, |\bar{Y}|)$. For simplicity we consider only the OPE of the primary fields V_1 and V_2 : this is possible if we restrict ourselves to the 4-point conformal blocks only, which we really do in this paper.

With the OPE, one can project the products $V_1(x_1)V_2(x_2)$ and $V_3(x_3)V_4(x_4)$ onto Verma modules given by $V_{\Delta, \bar{\Delta}}$ and $V_{\Delta', \bar{\Delta}'}$ correspondingly, and since the pair correlation function of any fields is non-zero iff the both fields have the same conformal dimension, one finally obtains (3.1), and the conformal block represents a chiral part of the correlation function with specified intermediate dimension. It can be defined completely within the chiral algebra [51].

In terms of the chiral algebra one defines the conformal block through a chiral correlator:

$$B_{\Delta}(x) = \left\langle V_{\Delta_1}(x_1) V_{\Delta_2}(x_2) \right|_{\Delta} V_{\Delta}(x_3) V_{\Delta}(x_4) \rangle \quad (3.5)$$

where hereafter we denote the corresponding chiral primary fields with the same letter V , and the chiral OPE looks like

$$V_{\Delta_1}(x_1) V_{\Delta_2}(x_2) \xrightarrow{\Delta} (x_1 - x_2)^{\Delta - \Delta_1 - \Delta_2} \left(\sum_Y (x_1 - x_2)^{|Y|} \beta_Y(\Delta | \Delta_1, \Delta_2) \hat{L}_{-Y} V_{\Delta}(x_1) \right) \quad (3.6)$$

The coefficients β_Y are easily related with the three point functions: one suffices to consider the three point function $\Gamma_Y(\Delta_1, \Delta_2 | \Delta) = \langle V_{\Delta_1} V_{\Delta_2} | \hat{L}_{-Y} V_{\Delta} \rangle$ and use (3.6). Then, one immediately obtains for the three point function

$$\Gamma_Y(\Delta_1, \Delta_2 | \Delta) = \sum_{Y'} T_{Y, Y'}(\Delta) \beta_{Y'}(\Delta | \Delta_1, \Delta_2) \quad (3.7)$$

with the Shapovalov matrix

$$T_{Y, Y'}(\Delta) = \langle \hat{L}_{-Y} V_{\Delta} | \hat{L}_{-Y'} V_{\Delta} \rangle \quad (3.8)$$

Note that picking up a single term in the sum over Δ may lead to losing associativity of the product, which in simplest examples is guaranteed by the structure constants $C_{12}^{\Delta, \bar{\Delta}}$. For many models (when there are degenerate representations of the Virasoro algebra in the spectrum) they vanish for most values of $\Delta, \bar{\Delta}$. However, in these cases it is often enough to just impose selection rules on Δ . In any case, for the 4-point conformal blocks one can avoid using associativity. At this stage one gets

$$B_{\Delta}(\Delta_i | x) = \sum_{Y, Y'} x_{12}^{|Y|} x_{34}^{|Y'|} \beta_Y(\Delta | \Delta_1, \Delta_2) \beta_{Y'}(\Delta | \Delta_3, \Delta_4) \langle \hat{L}_{-Y} V_{\Delta} | \hat{L}_{-Y'} V_{\Delta} \rangle = \sum_{Y, Y'} \Gamma_Y(\Delta_1, \Delta_2 | \Delta) T_{Y, Y'}^{-1} \Gamma_{Y'}(\Delta_3, \Delta_4 | \Delta) \quad (3.9)$$

At last, in order to calculate the 3-point functions, one needs to use properties of the chiral algebra:

- consistency of the Virasoro algebra with the scalar product $\langle \dots \rangle$:

$$\langle \hat{L}_Y V | V' \rangle = \langle V | \hat{L}_{-Y} V' \rangle \quad (3.10)$$

for arbitrary operators V and V' (not necessarily primary). We normalize the primaries so that $\langle V_{\Delta} | V_{\Delta} \rangle = 1$. In fact, just this identity implies that the scalar product in (3.8) is equal to the Shapovalov matrix

$$\langle \hat{L}_{-Y} V_{\Delta} | \hat{L}_{-Y'} V_{\Delta} \rangle = \langle V_{\Delta} | \hat{L}_Y \hat{L}_{-Y'} V_{\Delta} \rangle, \quad (3.11)$$

in particular it vanishes for Young diagrams of differing sizes $|Y| \neq |\bar{Y}|$, what is important for the projective invariance of the correlator, i.e. for collecting the four coordinates x_1, \dots, x_4 into a single double ratio $x = \frac{(x_2-x_1)(x_3-x_4)}{(x_3-x_1)(x_2-x_4)}$, which allows one to put $x_1 = 0$, $x_2 = x$, $x_3 = 1$, $x_4 = \infty$.

- The comultiplication for the Virasoro algebra [51], which is a direct consequence of the Ward identities [1]:

$$\hat{L}_n \left(V_1(0) V_2(x) \right) = \left(\sum_{k=0}^{\infty} x^{n+1-k} \binom{n+1}{k} \hat{L}_{k-1} V_1(0) \right) V_2(x) + V_1(0) \hat{L}_n V_2(x) \quad (3.12)$$

Applying these postulates, one immediately obtains [1, 52, 53] that for $Y = \{y_1 \geq y_2 \geq \dots\}$

$$\Gamma_Y(\Delta_1, \Delta_2 | \Delta) = \prod_i \left(\Delta + y_i \Delta_2 - \Delta_1 - \sum_{j < i} y_j \right) \quad (3.13)$$

3.2 Chain-vectors

Alternatively, one can use a very effective representation of B_k in (1.1) via scalar products of constituents of the distinguished *chain-vectors* [1]

$$B_k = \left\langle k, \Delta_3, \Delta_4 \mid k, \Delta_1, \Delta_2 \right\rangle \quad (3.14)$$

which depend also on Δ and c and satisfy simple defining recurrence relations

$$\begin{aligned} \hat{L}_n \mid k, \Delta_1, \Delta_2 \rangle &= \left(k - n + \Delta + n \Delta_2 - \Delta_1 \right) \mid k - n, \Delta_1, \Delta_2 \rangle, \quad 0 < n \leq k, \\ \hat{L}_0 \mid k, \Delta_1, \Delta_2 \rangle &= \left(k + \Delta \right) \mid k, \Delta_1, \Delta_2 \rangle \end{aligned} \quad (3.15)$$

The possibility for such recurrence relations to unambiguously define a chain of such states within Verma module is a prominent feature of Virasoro algebra, which holds also for $\widehat{U(1)}$ but not for higher W_N algebras, beginning from W_3 . In fact, the chain vector is basically nothing but the projection of the operator product expansion of two primaries with conformal dimensions Δ_1, Δ_2 onto a third one with dimension Δ :

$$\begin{aligned} \hat{V}_{\Delta_1}(0) \hat{V}_{\Delta_2}(x) &\xrightarrow{\Delta} x^{\Delta - \Delta_1 - \Delta_2} \sum_Y x^{|Y|} \beta_Y(\Delta | \Delta_1, \Delta_2) \hat{L}_{-Y} \hat{V}_{\Delta}(0), \\ \mid k, \Delta_1, \Delta_2 \rangle &= \sum_{|Y|=k} \beta_Y(\Delta | \Delta_1, \Delta_2) \hat{L}_{-Y} \hat{V}_{\Delta}(0) \end{aligned} \quad (3.16)$$

i.e.

$$B_k = \sum_{|Y_1|=|Y_2|=k} \beta_Y(\Delta | \Delta_1, \Delta_2) \beta_Y(\Delta | \Delta_3, \Delta_4) \left\langle \hat{V}_{\Delta} \hat{L}_{Y_1} \mid \hat{L}_{-Y_2} \hat{V}_{\Delta}(0) \right\rangle \quad (3.17)$$

Importance of the chain vectors becomes especially clear within the AGT conjecture, since projectors of the chain vectors for product of the Virasoro and Heisenberg algebras on peculiar states in the Verma module associated with the generalized Jack polynomials reproduce the Nekrasov functions,

$$N_{Y_1, Y_2} = \left\langle k, \Delta_3, \Delta_4 \mid J_{Y_1, Y_2} \right\rangle \left\langle J_{Y_1, Y_2} \mid k, \Delta_1, \Delta_2 \right\rangle \quad (3.18)$$

(here pairs of Young diagrams emerge due to the product of the Virasoro and Heisenberg algebras) and they have extra poles not present in the scalar products (3.14). For various realizations of this idea see [19].²

²For instance, at the level one:

$$\mid J_{0,1} \rangle = \left(\hat{L}_{-1} + (Q + 2a) \hat{a}_{-1} \right) \mid a \rangle \quad \mid J_{1,0} \rangle = \left(\hat{L}_{-1} + (Q - 2a) \hat{a}_{-1} \right) \mid a \rangle \quad (3.19)$$

i.e.

$$\hat{L}_{-1} \mid a \rangle = -\frac{1}{4a} \left((Q - 2a) \mid J_{0,1} \rangle - (Q + 2a) \mid J_{1,0} \rangle \right) \quad \hat{a}_{-1} \mid a \rangle = \frac{1}{4a} \left(\mid J_{0,1} \rangle - \mid J_{1,0} \rangle \right) \quad (3.20)$$

3.3 Explicit expressions

In result we get for the first several coefficients B_k in

$$B(x) = 1 + \sum_{k=1}^{\infty} B_k x^k \quad (3.21)$$

the following explicit expressions:

$$\begin{aligned} B_1 &= \frac{(\Delta + \Delta_2 - \Delta_1)(\Delta + \Delta_3 - \Delta_4)}{2\Delta} \\ \mathcal{B}_{\Delta}^{(2)} &= \frac{(\Delta + \Delta_2 - \Delta_1)(\Delta + \Delta_2 - \Delta_1 + 1)(\Delta + \Delta_3 - \Delta_4)(\Delta + \Delta_3 - \Delta_4 + 1)}{4\Delta(2\Delta + 1)} + \\ &+ \frac{[(\Delta_1 + \Delta_2)(2\Delta + 1) + \Delta(\Delta - 1) - 3(\Delta_2 - \Delta_1)^2][(\Delta_3 + \Delta_4)(2\Delta + 1) + \Delta(\Delta - 1) - 3(\Delta_3 - \Delta_4)^2]}{2(2\Delta + 1)(2\Delta(8\Delta - 5) + (2\Delta + 1)c)} \\ &\dots \end{aligned} \quad (3.22)$$

Denominators of these expressions,

$$\begin{aligned} K_2 &= 4\Delta(16\Delta^2 - 10\Delta + 2c\Delta + c), \\ K_3 &= 6(3\Delta^2 + c\Delta - 7\Delta + c + 2) \cdot K_2, \\ K_4 &= 4(8\Delta + c - 1)(16\Delta^2 - 82\Delta + 10c\Delta + 15c + 66) \cdot K_3, \\ &\dots \end{aligned} \quad (3.23)$$

at generic values of c possess only simple zeroes, which coincide with zeroes of the Kac determinants KD_n (i.e. determinants of Shapovalov matrices), though these latter sometimes have non-zero multiplicities. These extra (multiplicity of) zeroes of the Kac determinants cancel against zeroes of the numerators in expressions for the 4-point conformal blocks, e.g.

$$\begin{aligned} KD_3 &= 2\Delta \cdot K_3, \\ KD_4 &= 4\Delta^2(16\Delta^2 - 10\Delta + 2c\Delta + c) \cdot K_4 \end{aligned} \quad (3.24)$$

Since the cancelation takes place at arbitrary points of the moduli space \mathcal{M} (i.e. for arbitrary dimensions and central charges), these zeroes play no role in further our considerations, and we sometimes call the reduced quantities (3.24) the Kac determinants assuming that this should not cause any confusion.

3.4 Coefficients B_k from Dotsenko-Fateev representation of [17]

Let us make the change of variables

$$\Delta_{\mu} = \alpha_{\mu}(\alpha_{\mu} - Q), c = 1 - 6Q^2, Q = b - \frac{1}{b} \quad (3.25)$$

with α_i constrained by

$$\begin{aligned} \alpha - \alpha_1 - \alpha_2 &= bN_1, \\ Q - \alpha - \alpha_3 - \alpha_4 &= bN_2 \end{aligned} \quad (3.26)$$

In fact, one can choose in (3.27) $Q - \alpha_i$ instead of any α_i , since there is a symmetry in the theory w.r.t. this operation (in particular, (3.26) remains unchanged under this transformation).

With this change of variables (3.26) $B_k(\Delta_1, \dots, \Delta_4, \Delta, c)$ turns into rational functions of $J_k(\alpha_1, \alpha_2 + \alpha_3, b, N_1, N_2)$,

$$B_k = J_k \quad (3.27)$$

which at integer non-negative values of N_1 and N_2 coincide with the values of Selberg-Kadell[54] integrals, N_1 times between 0 and x and N_2 times between 0 and 1. This fact [17, 55] can be interpreted as Dotsenko-Fateev like representation of conformal blocks [46] via conformal matrix model of [56, 16].

In more detail,

$$J_k = \frac{Z(v)}{Z(0)}, \quad (3.29)$$

$$Z(v) = \int \prod_{a < a'} (v_a - v_{a'})^{2b^2} \prod_a v_a^{2\alpha_1 b} (x - v_a)^{2\alpha_2 b} (1 - v_a)^{2\alpha_3 b} dv_a$$

and N_1 integrations here runs 0 to x , while N_2 integrations goes from 0 to 1. Note that the integrals are not obligatory around the *closed* contours: for irrational products $\alpha_i \alpha_j$ these are not so easy to define. In other words, following [17] we define the integrals in the same way as the archetypical B -function integral

$$\int_0^1 z^{a-1} (1-z)^{b-1} dz = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

is defined. The new point is analytical continuation in N_1 and N_2 : since B_k are *rational* functions of these variables, continuation is straightforward and unambiguous.

These formulas can also be straightforwardly q -deformed [21, 22], this generalization is related to $5d$ gauge theories, to q -Virasoro algebras, to MacDonald polynomials and DAHA.

3.5 General structure of B_k

The poles of B_k (Kac zeroes) have a very simple form in the Dotsenko-Fateev parametrization: they occur at all integers $|m| > 0$, $|n| > 0$ such that $mn \leq k$ at the points

$$\alpha_{m,n} = \frac{1}{2} \left(\frac{n-1}{b} - (m-1)b \right) \quad (3.30)$$

It follows that once a pole appears at some B_k , it also happens at all $B_{l>k}$. Such a singularity in the conformal block at level k can be removed by a proper choice of the external dimensions Δ_i so that the theory would still make sense. Surprisingly enough, once the pole disappears in this way at the level k , it simultaneously disappears at all higher levels. In other words, not only the Kac zeroes, but also the zeroes of the numerators have a nested structure. In the next sections we study this phenomenon in detail, in particular, we observe that in conformal theories with rational values of the central charge there emerge multiple Kac zeroes, which sometimes still cancel out with multiple zeroes of the numerator, but sometimes not.

Our goal in the next sections is to look for a non-perturbative parameters in the conformal block, i.e. we look for conformal blocks which are not uniquely defined *functions* of x at given Δ_i , Δ , c . One immediate example is provided in the point $\Delta_i = 1/16$, $\Delta = 1/4$, $c = 1$ (see s.4): this is the case of Ashkin-Teller model solved by Al.Zamolodchikov [32]. On the other hand, at the same point there is another conformal block, given by an elliptic integral which follows from formulas in [17]. The existence of two different solutions may imply that this would provide us with needed non-perturbative parameter. However, as we explain later in s.5, this is not the case: the ambiguity is just due to different possibilities of approaching the singularity, since the Ashkin-Teller model is located in the moduli space exactly at the Kac zero.

Another possibility to look for non-perturbative parameters could be in the cases when some of the external dimensions correspond to a degenerate vector. If this vector is degenerate at level k and one specially matches the intermediate dimension, there is a differential equation of order k in x for the conformal block. This equations has generally k independent solutions and, hence, the full answer would be a linear combination of these, depending on $k-1$ arbitrary constants. However, these constants can not be associated with non-perturbative ambiguity in the definition of the conformal block, since each of these k solutions correspond to exactly one conformal block fixed by a proper asymptotics! Moreover, the modular transformation $x \rightarrow 1-x$ transforms any of these conformal blocks through remaining $k-1$.

In fact, the absence of non-perturbative parameter in the second case is not surprising: we already mentioned that as a function of x the 4-point spherical conformal block is believed to be a function, ramified just at three points: 0, 1, ∞ , with no essential singularities. Hence, the expansion series in x have finite radii of convergence, and in this sense they are somewhat different, say, from (2.2). However, the convergence in x is not uniform in the other parameters (Δ_i, Δ, c) , and this makes the entire function of all variables quite sophisticated, which we observe in the Ashkin-Teller case.

4 Non-perturbative parameters. First quest: Ashkin-Teller model

4.1 Elliptic integrals

Note that (3.29) is an integral representation of entire conformal block $\mathcal{B}(x)$, not of individual coefficients $J_k = B_k$ of its x -expansion. A natural idea could be to use such representation as an obvious candidate for a non-perturbative definition.

Of course, this idea raises a number of interesting questions – especially, about the analytical continuation in N and associated non-perturbative dependencies on intermediate dimensions, like Δ . This would open a way to study non-perturbative modular kernels [40, 41, 57, 59, 39] (which perturbatively are just Fourier transforms [58]).

However, before going deeper in that direction, it makes sense to look at this approach in a less controversial situation: at natural (positive integer) values of N_1 and N_2 . It deserves beginning from the simplest case of $c = 1$, i.e. $b = 1$. In this case there is a solution of (3.27) at $N_1 = 1$, $N_2 = 0$.

Even after that there is a further simplification: for a special choice of external dimensions, when the integral becomes elliptic.

For instance, at $\alpha_i = -1/4$ the integral (3.29) turns into just an ordinary elliptic integral

$$K(x) = \int \frac{dz}{y(z)} \quad (4.1)$$

with $y^2(z) = z(1-z)(z-x)$, which is elementary to analyze, both perturbatively and non-perturbatively.

$$\int_0^1 \frac{dz}{y(z)} = \sum c_k^2 x^k = 1 + \frac{1}{4}x + \frac{9}{64}x^2 + \dots \quad (4.2)$$

and

$$\begin{aligned} \mathcal{B}^{ell}(x) = (1-x)^{1/8} K(x) = & 1 + \frac{1}{8}x + \frac{7}{128}x^2 + \frac{33}{1024}x^3 + \frac{713}{32768}x^4 + \frac{4165}{262144}x^5 + \\ & + \frac{51205}{4194304}x^6 + \frac{326255}{33554432}x^7 + \frac{17078585}{2147483648}x^8 + \frac{114071265}{17179869184}x^9 + \dots \end{aligned} \quad (4.3)$$

The MAPLE command to generate this formula is

```
series((1-x)^(1/8)*EllipticK(sqrt(x))/Pi*2,x,10);
```

For non-perturbative analysis of integrals the best method is via the Picard-Fuchs equations. The Picard-Fuchs equation for

$$\Pi = \oint_C \frac{dz}{y(z)} = \oint_C \frac{dz}{\sqrt{z(1-z)(z-x)}} \quad (4.4)$$

along *any closed* contour C is

$$\left(x(1-x) \frac{\partial^2}{\partial x^2} + (1-2x) \frac{\partial}{\partial x} + \frac{1}{4} \right) \Pi = 0 \quad (4.5)$$

There are two solutions: one is $K(x)$ having the asymptotics 1 at small x , the other one, $K'(x)$ has the asymptotics $\log x$ [60]. We are definitely interested in the first case.

4.2 Zamolodchikov's formula

In [28] Al.Zamolodchikov suggested a wonderful formula for the non-perturbative conformal block at the slice where $\Delta_i = \frac{1}{16}$, $c = 1$:

$$\mathcal{B}_\Delta^{Zam}(x) = \mathcal{B}_\Delta \left(\Delta_i = \frac{1}{16}, c = 1 \mid x \right) = (1-x)^{-1/8} \frac{(16q/x)^\Delta}{\theta_{00}(q)} \quad (4.6)$$

where relation between x , considered as a ramification point, and elliptic parameter $q = e^{i\pi\tau}$ is given by

$$x = \frac{\theta_{10}^4}{\theta_{00}^4} = 16q \cdot \frac{(1+q^2+q^6+\dots)^4}{(1+2q+2q^4+\dots)^4} = 16q - 128q^2 + \dots \quad (4.7)$$

and the theta-constants

$$\begin{aligned}
\theta_{00}(q) &= \theta_{00}(0|\tau) = 1 + 2 \sum_{n=1}^{\infty} q^{n^2} = \prod_{m=1}^{\infty} (1 - q^{2m})(1 + q^{2m-1})^2 \\
\theta_{01}(q) &= \theta_{01}(0|\tau) = 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{n^2} = \prod_{m=1}^{\infty} (1 - q^{2m})(1 - q^{2m-1})^2 \\
\theta_{10}(q) &= \theta_{10}(0|\tau) = 2q^{1/4} \sum_{n=0}^{\infty} q^{n(n+1)} = 2q^{1/4} \prod_{m=1}^{\infty} (1 - q^{2m})(1 + q^{2m})^2 \\
\theta'_{11}(q) &= \theta_{00}\theta_{01}\theta_{10} = 2q^{1/4} \prod_{m=1}^{\infty} (1 - q^{2m})^3 = \eta^3(q)
\end{aligned} \tag{4.8}$$

Equation (4.7) relates the modular transformations of x , generated by $x \rightarrow 1 - x$ and $x \rightarrow -1/x + 1$ to the modular transformations of theta constants generated by $\theta_{\epsilon,\delta}(0| -1/\tau) = (-i)^{\epsilon\delta} \sqrt{-i\tau} \theta_{\delta,\epsilon}(0|\tau)$ and $\theta_{\epsilon,\delta}(0|\tau + 1) = e^{\pi i/4\epsilon} \theta_{\epsilon,\delta+1-\epsilon}(0|\tau)$, where the characteristics of the θ -functions ϵ and δ are understood as taken by modulo 2. The crucial role in this relation is played by the Riemann identity $\theta_{00}^4(0|\tau) = \theta_{01}^4(0|\tau) + \theta_{10}^4(0|\tau)$.

At the Ashkin-Teller point μ_{AT} , where additionally $\Delta = \frac{1}{4}$,

$$(1-x)^{1/8} \mathcal{B}_{1/4}^{Zam} = \frac{(16q/x)^{1/4}}{\theta_{00}} = \frac{2q^{1/4}}{\theta_{10}} = \frac{1}{1 + q^2 + q^6 + \dots} \tag{4.9}$$

and perturbative expansion near in powers of x is

$$\begin{aligned}
B_{1/4}^{Zam}(x) &= \left(1 + \frac{1}{8}x + \frac{9}{128}x^2 + \dots\right) \left(1 - \frac{1}{256}x^2 + \dots\right) = 1 + \frac{1}{8}x + \frac{17}{256}x^2 + \frac{93}{2048}x^3 + \frac{2269}{65536}x^4 + \\
&+ \frac{14705}{524288}x^5 + \frac{198109}{8388608}x^6 + \frac{1370655}{67108864}x^7 + \frac{77366631}{4294967296}x^8 + \frac{554104463}{34359738368}x^9 + \dots
\end{aligned} \tag{4.10}$$

The MAPLE command to generate this formula is

```

q:=x->EllipticNome(x);
A:=x->(1-x)^(-1/8)*(16*q(sqrt(x))/x)^(1/4)/JacobiTheta3(0,q(sqrt(x)));
series(A(x),x,10);

```

4.3 Intersection of physical and elliptic slices

4.3.1 The problem

Now we can compare the two expressions (4.3) and (4.10). Both correspond to the Ashkin-Teller point

$$\mu_{AT} : \quad \Delta_i = \frac{1}{16}, \quad \Delta = \frac{1}{4}, \quad c = 1 \tag{4.11}$$

in the moduli space \mathcal{M} , both are expansions of the well defined and well known functions of x , but they are different already in the second terms of the x -expansion!

What does this mean? How could we get two different expressions for B_2 at the given point?

The reason is that μ_{AT} actually lies on the divisor of B_2 . In order to obtain the second coefficient of the x -expansion from (3.23) one needs to resolve the singularity $\frac{0}{0}$, and the resolution is ambiguous. Of course, this is a usual situation for a function of many variables (even rational, like B_k), still this causes additional ambiguities in the conformal block: it is not uniquely defined by the conformal dimensions at the Kac zeroes.

Our immediate goal in this situation is to study the vicinity of μ_{AT} and see how the two expressions (4.3) and (4.10) emerge from a single (3.23). At the next step we should look at other coefficients B_k and at other interesting points on the divisor.

4.3.2 B_2 in the vicinity of μ_{AT}

Thus, we substitute

$$\Delta_i = \frac{1}{16} + \epsilon\delta_i, \quad \Delta = \frac{1}{4} + \epsilon\delta, \quad c = 1 + \epsilon\sigma \tag{4.12}$$

into expression (3.22), (3.23), ... for the coefficients of the x -expansion of conformal block and look at what happens at small values of ϵ .

For B_1 we get nothing interesting:

$$B_1^{AT} = \frac{1}{8} + O(\epsilon) \quad (4.13)$$

and this is exactly what is needed in *both* (4.3) and (4.10)

However, for B_2 the situation gets far more interesting:

$$B_2^{AT} = \frac{\frac{25}{256}\sigma\epsilon + O(\epsilon^2)}{\frac{3}{2}\sigma\epsilon + O(\epsilon^2)} \stackrel{?}{=} \frac{25}{384} + O(\epsilon) \quad (4.14)$$

This is still another rational number *different* from those in both (4.3) and (4.10).

However, let us still look at the next order in ϵ :

$$B_2^{AT} = \frac{\frac{25}{256}\sigma\epsilon + \rho\epsilon^2 + O(\epsilon^3)}{\frac{3}{2}\sigma\epsilon + (16\delta^2 + 8\sigma\delta)\epsilon^2 + O(\epsilon^3)} \quad (4.15)$$

where

$$\rho = \frac{1}{16} \left(17\delta^2 - 2\delta(\delta_1 + \delta_2 + \delta_3 + \delta_4) + 12(\delta_1 + \delta_2)(\delta_3 + \delta_4) \right) + \frac{15}{32}\sigma \left(2\delta - \delta_1 + \delta_2 + \delta_3 - \delta_4 \right) \quad (4.16)$$

Now one can see what happens. In both cases (4.3) and (4.10) one approach the point μ_{AT} keeping $\sigma = 0$. Then

$$B_2^{AT}(c=1) = \frac{1}{256} \left(17 - 2\frac{\delta_1 + \delta_2 + \delta_3 + \delta_4}{\delta} + 12\frac{(\delta_1 + \delta_2)(\delta_3 + \delta_4)}{\delta^2} \right) + O(\epsilon) \quad (4.17)$$

Zamolodchikov's formula (4.10) refers to the physical slice, when the theory and observables (external dimensions Δ_i) are fixed: thus, $\sigma = 0$ and $\delta_1 = \dots = \delta_4 = 0$, and only the intermediate dimension *could* vary. In other words, only δ is *imagined* to be non-vanishing. When we approach μ_{AT} from this special direction we get

$$\lim_{\delta \rightarrow 0} B_2^{AT} \left(\Delta_i = \frac{1}{16}, c=1 \right) = \frac{17}{256} \quad (4.18)$$

i.e. the answer is (4.10).

For the elliptic integral (4.3) the situation is absolutely different. What is fixed in this case is the number of integrations, $N_1 = 1$, $N_2 = 0$. According to (3.27) this implies that we approach μ_{AT} from a very different direction, where

$$\begin{aligned} \sigma &= 0, \\ \delta + 2(\delta_1 + \delta_2) &= 0, \\ \delta - 2(\delta_3 + \delta_4) &= 0 \end{aligned} \quad (4.19)$$

Note that (3.27) is written in terms of α -parameters, not dimensions, and at $c=1$ our $\delta_i = \frac{\alpha_i - \alpha_i^{AT}}{2\epsilon\sqrt{\Delta_i}}$, hence, the additional coefficient $2 = \frac{\sqrt{1/4}}{\sqrt{1/16}}$ in (4.19). Then (4.17) gives

$$\lim_{\delta \rightarrow 0} B_2^{AT} \left((4.19), c=1 \right) = \frac{1}{256} \left(17 - \frac{12}{2^2} \right) = \frac{7}{128} \quad (4.20)$$

i.e. exactly what is needed for (4.3). Note that (4.19) imposes only two constraints, but this turns to be enough to provide an unambiguous limit in (4.17).

In fact, it is both convenient and natural to put the central charge and the dimensions on equal footing. If we parameterize $c = 1 - 6(b - 1/b)^2$, see (3.26), and use $b = 1 + \epsilon\eta$ instead of $c = 1 + \epsilon\sigma$, deviations from the AT point will be entirely of the order ϵ^2 in the numerator and denominator. In other words,

$$\sigma = -24\eta^2\epsilon \quad (4.21)$$

and this is the resolution of singularity that we use in the rest of this section. In particular, in this parametrization

$$B_2^{AT} = \frac{\frac{17}{16}\delta^2 - \frac{75}{32}\eta^2 - \frac{1}{8}\delta(\delta_1 + \delta_2 + \delta_3 + \delta_4) + \frac{3}{4}(\delta_1 + \delta_2)(\delta_3 + \delta_4) + O(\epsilon)}{16\delta^2 - 36\eta^2 + O(\epsilon)} \quad (4.22)$$

where we omitted the common overall factors ϵ^2 in the numerator and denominator.

4.3.3 Other B_k : universality and the germ of conformal block at μ_{AT}

If one now makes the same substitution in B_3 , one again obtains the double zeroes at μ_{AT} in the numerator and denominator, and

$$B_3^{AT} = \frac{\frac{7533}{1024}\delta^2 - \frac{32805}{2048}\eta^2 - \frac{729}{512}\delta(\delta_1 + \delta_2 + \delta_3 + \delta_4) + \frac{2187}{256}(\delta_1 + \delta_2)(\delta_3 + \delta_4) + O(\epsilon)}{162\delta^2 - \frac{729}{2}\eta^2 + O(\epsilon)} \quad (4.23)$$

The numbers can look ugly, however they are in fact just the same as in (4.22):

$$B_3^{AT} = -\frac{15}{512} + \frac{9}{8} \cdot B_2^{AT} + O(\epsilon) \quad (4.24)$$

This means that we do not need to perform any *independent* calculation in the third terms in (4.3) and (4.10): inter-relation between these two cases is fully fixed at the level of the second coefficient.

Indeed, for B_4 the same property persists for higher B_k :

$$\begin{aligned} B_1 &= \frac{1}{8}, & B_2 &= \frac{17-r}{256}, & B_3 &= \frac{93-9r}{2^{12}}, & B_4 &= \frac{2269-281r}{2^{16}}, \\ B_5 &= \frac{14705-2125r}{2^{19}} = \frac{5 \cdot 17}{2^{19}} \cdot (173-25r), & \dots \end{aligned} \quad (4.25)$$

where

$$r \equiv \frac{-3\eta^2 + 8\delta(\delta_1 + \delta_2 + \delta_3 + \delta_4) - 48(\delta_1 + \delta_2)(\delta_3 + \delta_4)}{4\delta^2 - 9\eta^2} \quad (4.26)$$

It is natural to *assume* that this remains true in general:

$$B_k = B_k^{Zam} + \frac{r}{3} (B_k^{ell} - B_k^{Zam}), \quad k < 6 \quad (4.27)$$

However, this is actually true only for $k < 6$. Indeed, at level 6 there Kac zero at $\Delta = 1/4$ becomes of the fourth order: $(n-m)^2/4 = 1/4$ when $n = 2$, $m = 1$, i.e. at level $n \cdot m = 2$; when $n = 3$, $m = 2$, i.e. at level $n \cdot m = 6$ etc. The numerator still cancels this multiple zero at the Ashkin-Teller point, however, the ambiguity becomes a ratio of two quartic polynomials of δ_i, δ, η . For $6 \leq k < 12$ we have:

$$B_k = B_k^{Zam} + \frac{r}{3} (B_k^{ell} - B_k^{Zam}) + \frac{r_2}{3 \cdot 2^{11}} C_k, \quad 6 \leq k < 12 \quad (4.28)$$

with

$$C_{k < 6} = 0, \quad C_6 = 1, \quad C_7 = \frac{25}{8}, \quad C_8 = \frac{1577}{800} C_7, \quad \dots \quad (4.29)$$

where

$$r_2 = \frac{P_4(\eta, \delta, \delta_i)}{(4\delta^2 - 9\eta^2)(4\delta^2 - 25\eta^2)} \quad (4.30)$$

and

$$\begin{aligned} P_4(\eta, \delta, \delta_i) &= -60\eta^4 + \eta^2 \left(15\delta^2 + 86\delta(\delta_1 + \delta_2 + \delta_3 + \delta_4) + 120(\delta_1 + \delta_2)^2 + 120(\delta_3 + \delta_4)^2 - 660(\delta_1 + \delta_2)(\delta_3 + \delta_4) \right) - \\ &\quad - 8\delta^3(\delta_1 + \delta_2 + \delta_3 + \delta_4) + 48\delta^2 \left(3(\delta_1 + \delta_2)(\delta_3 + \delta_4) - (\delta_1 + \delta_2)^2 - (\delta_3 + \delta_4)^2 \right) - \\ &\quad - 192\delta(\delta_1 + \delta_2)(\delta_3 + \delta_4)(\delta_1 + \delta_2 + \delta_3 + \delta_4) + 960(\delta_1 + \delta_2)^2(\delta_3 + \delta_4)^2 \end{aligned} \quad (4.31)$$

The moral of this story, is that the behavior of conformal block in the vicinity of the point μ_{AT} is *universal*: does not depend on the order k of the x -expansion. This opens a possibility to suggest a formula for the *germ* of conformal block at μ_{AT} (the next r_3 emerge at level 12, when the Kac zero gets multiplicity 6):

$$\begin{aligned} B(x) &= B^{Zam}(x) + \frac{r}{3} (B^{ell}(x) - B^{Zam}(x)) + \\ &\quad + \frac{r_2}{3} \cdot \frac{x^6}{2^{11}} \left(1 + \frac{25}{8}x + \frac{1577}{256}x^2 + \frac{20141}{2048}x^3 + \frac{911193}{65536}x^4 + \frac{9549597}{524288}x^5 + \dots \right) + x^{12}B_{12} + \dots + O(\epsilon) \end{aligned}$$

(4.32)

Zamolodchikov's expansion corresponds to all external $\delta_i = 0$, while the elliptic locus in the vicinity of the AT point is a union of several hyperplanes, each defined by two conditions:

$$\sum_{i=1}^4 \delta_i = -\frac{3}{2}\eta, \quad \delta = 2(\delta_3 + \delta_4) + \eta \quad (4.33)$$

or

$$\sum_{i=1}^4 \delta_i = \frac{3}{2}\eta, \quad \delta + 2(\delta_3 + \delta_4) = 2\eta \quad (4.34)$$

4.4 Chain vectors

As an alternative to the above technique, one can perform an analysis of the singularity locus in terms of the representation of B_k in (1.1) via the chain-vectors (s.3.2). For instance, at the Ashkin-Teller point the leading behaviour of the chain-vectors is given by

$$\begin{aligned} \beta_2 &= -\frac{\zeta}{\xi} + O(\xi) \\ \beta_{11} &= \frac{\zeta}{\xi} + O(\xi) \\ \beta_{21} &= -\frac{\zeta}{2\xi} + O(\xi) \\ \beta_{111} &= \frac{\zeta}{2\xi} + O(\xi) \\ \beta_3 &= 0 \end{aligned} \quad (4.35)$$

where

$$\zeta = \frac{\delta - 6\delta_1 - 6\delta_2}{2(32\delta^2 + 3\eta)} \quad (4.36)$$

and ξ is a distance from the singularity locus. Thus, the chain-vectors are non-zero vectors nearby the locus, but they must have zero norm in the leading order, since their norm (which is equal to the conformal block (3.14)) is finite on the locus. Indeed, the singularities cancel in the conformal block, because of degeneracy of the Shapovalov matrix (since the singularity locus is located in zeroes of determinant of the Shapovalov matrix)

$$\left(\sum_{|Y|=k} \text{sing}(\beta_Y) \right)^2 = 0 \quad (4.37)$$

5 Perturbative conformal block in the vicinity of Kac divisor

5.1 Poles of B_k and their nested structure

We already discussed in s.3 and demonstrated in the manifest Ashkin-Teller example in s.4 that

- if a zero appears in the Kac determinant K_k , it persists in all higher K_k with $k \geq m$. The Kac determinants depend only on the intermediate dimension Δ and the central charge c , and in the α -parametrization (3.26) the zeroes are actually at the points (3.30). At such points all the coefficients B_k with $k \geq m$ are singular.
- However, one can adjust external dimensions Δ_i so that the numerator in B_m also vanishes. What happens is that then it also vanishes in the numerators of all higher B_k with $k \geq m$.

For example, a Kac zero at the third level is at

$$\alpha_{1,3} = -b \quad (5.1)$$

The numerator of B_3 at this zero is

$$X = \frac{X_{12}X_{34}}{24b^5(b^2 - 1)(3b^2 - 1)(4b^4 - 1)} \quad (5.2)$$

with

$$X_{12} = (\alpha_1 - \alpha_2)(b^2 - 1 - b\alpha_1 - b\alpha_2)(2b^2 - 1 - b\alpha_1 - b\alpha_2)(b + \alpha_1 - \alpha_2)(b - \alpha_1 + \alpha_2)(1 + b\alpha_1 + b\alpha_2) \quad (5.3)$$

The numerator of B_4 is the same X , multiplied by

$$\frac{(2b^3 + 2b + \alpha_2^2 b - \alpha_2 b^2 + \alpha_2 - \alpha_1^2 b + \alpha_1 b^2 - \alpha_1)(2b^3 + 2b + \alpha_3^2 b - \alpha_3 b^2 + \alpha_3 - \alpha_4^2 b + \alpha_4 b^2 - \alpha_4)}{4b^2(b^2 + 1)} \quad (5.4)$$

This demonstrates that at the Kac zero of level $m = 3$ the zero of the numerator of B_m (the zero of X) remains a zero of the higher B_k , e.g. of B_4 .

Once again, not only the zero loci $V(K_k)$ of Kac determinants are nested,

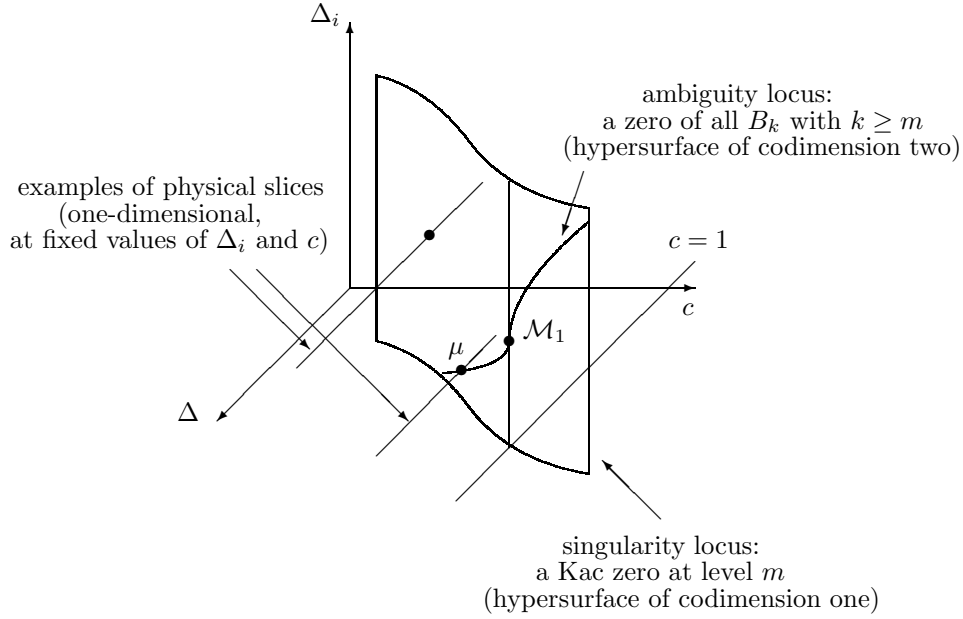
$$V(K_k) \subset V(K_l) \quad \text{or} \quad K_l \vdots K_k \quad \text{for} \quad l > k \quad (5.5)$$

the same is true for the *numerators* of B_k , provided they are restricted to $V(K_k)$: for $\mathcal{V}_k = \left\{ \text{zeroes of } B_k \Big|_{V(K_k)} \right\}$ we have

$$\boxed{\mathcal{V}_k \subset \mathcal{V}_l \quad \text{for} \quad k < l} \quad (5.6)$$

5.2 Coefficients B_k at intersection of zeroes of numerators and denominators

At generic point of $\bigcup_k V(K_k)$ the coefficients of conformal block are singular (have poles), thus this union form a *singularity locus* in the moduli space \mathcal{M} and it has codimension one. However, there is a codimension one hypersurface within the singularity locus (thus it has codimension two in \mathcal{M}), where the numerators are also vanishing. And at these points we have an ambiguity of the type 0/0, the value of the coefficients depending on the direction from which one approaches such point in \mathcal{M} . It is natural to name this codimension-two hypersurface *the ambiguity locus*. Our next goal is to describe behavior of the entire conformal block, not just of its particular coefficients B_k , at this locus.



Of interest for us in this section will be the points $\mu \in \mathcal{A} \subset \mathcal{M}$, lying at the ambiguity locus \mathcal{A} . The thing is that most of interesting well known examples of conformal models, including Ashkin-Teller and minimal models are of this type. As we already saw in discussion of the vicinity of the point $\mu_{AT} \in \mathcal{M}_1 \subset \mathcal{A} \subset \mathcal{M}$, all the coefficients B_k can behave similarly near this locus, thus their common properties are inherited by the entire conformal block.

As before, at the vicinity of a point $\mu \in \mathcal{A}$, $\mu = \{\Delta_i^{(0)}, \Delta^{(0)}, c^{(0)}\}$ we put $\Delta_i = \Delta_i^{(0)} + \epsilon \delta_i$, $\Delta = \Delta^{(0)} + \epsilon \delta$, $c = c^{(0)} + \epsilon \sigma$ or $b = b^{(0)} + \epsilon \eta$ with a 6-vector $\xi_I = (\delta, \delta_1, \dots, \delta_4, \eta)$, and it turns out that

$$B_k = b_k^{(0)} + d_k \cdot \frac{u_I \xi_I + O(\xi^2)}{v_I \xi_I + O(\xi^2)} \quad (5.7)$$

if the zero is simple,

$$B_k = b_k^{(0)} + d_k \cdot \frac{U_{IJ}\xi_I\xi_J + O(\xi^3)}{V_{IJ}\xi_I\xi_J + O(\xi^3)} \quad (5.8)$$

if it is double zero etc. It often happens (as in the previous section) that

$$\boxed{\text{linear/quadratic forms } u, v \text{ and } U, V \text{ are independent of } k} \quad (5.9)$$

Note that the modified parametrization (4.21) is essential only if $c^{(0)} = 1$, otherwise, σ and η are related linearly without any additional damping factor ϵ .

Thus at the critical point we have both an ambiguity (the limit depends on the choice of direction to approach the point) and a universality, expressed by (5.6) and (5.9).

In the rest of this section we consider a few more examples of (5.6) and (5.9).

5.3 Phenomenon 1: nested structure of zeroes. $c = 1$ example

At $c = 1$ all zeroes of the Kac determinants are doubled ($\Delta_0 = \alpha^2$ is itself a full square):

$$\begin{aligned} K_2 &= 4\Delta(4\Delta - 1)^2, \\ K_3 &= 72\Delta(4\Delta - 1)^2(\Delta - 1)^2, \\ K_4 &= 2304\Delta^2(4\Delta - 1)^2(\Delta - 1)^2(4\Delta - 9)^2, \end{aligned} \quad (5.10)$$

...

The numerators of the coefficients B_k at $c = 1$ are nothing special. For instance, if one takes all the external dimensions equal to each other, $\Delta_1 = \dots = \Delta_4 = \Delta_e$, the numerators of B_2 and B_3 are respectively

$$NB_2 = \Delta \left(\Delta - 6\Delta^2 + 9\Delta^3 + 8\Delta^4 + 16\Delta_e^2\Delta - 8\Delta_e\Delta + 8\Delta_e\Delta^2 + 8\Delta_e^2 \right) \quad (5.11)$$

and

$$NB_3 = 3\Delta(\Delta + 2)(\Delta - 1)^2 \left(8\Delta^4 + 19\Delta^3 + 24\Delta_e\Delta^2 - 11\Delta^2 + 48\Delta_e^2\Delta - 24\Delta_e\Delta + 2\Delta + 24\Delta_e^2 \right) \quad (5.12)$$

Things, however, change considerably if one looks at these numerators at the Kac zeroes. For example, at $\Delta = \alpha_{2,1}^2 = \Delta_{2,1}(c = 1) = \Delta_{1,2}(c = 1) = \frac{1}{4}$,

$$NB_2 = \frac{3}{256} \left(1 - 8(\Delta_1 + \Delta_2) + 16(\Delta_1^2 + \Delta_2^2) - 32\Delta_1\Delta_2 \right) \left(1 - 8(\Delta_3 + \Delta_4) + 16(\Delta_3^2 + \Delta_4^2) - 32\Delta_3\Delta_4 \right) \equiv \frac{3}{256} nb_2,$$

$$NB_3 = -\frac{27}{16384} \left(9 + 4(\Delta_1 - \Delta_2) \right) \left(9 + 4(\Delta_3 - \Delta_4) \right) \cdot nb_2,$$

$$NB_4 \sim nb_2,$$

...

$$NB_k \sim nb_2 \quad \text{for } k \geq 2$$

$$(5.13)$$

This is how (5.6) is realized in this case: any zero of NB_2 remains zero of higher NB_k .

Let us now look at the vicinity of a double zero of this NB_2 . In this case $\Delta_1 = \Delta_2 \pm \sqrt{\Delta_2} + 1/4$ $\Delta_4 = \Delta_3 \pm \sqrt{\Delta_3} + 1/4$ and one has (choosing for the sake of definiteness both signs plus):

$$B_2 = \frac{\sqrt{\Delta_2\Delta_3}(4\sqrt{\Delta_2\Delta_3} - 2\sqrt{\Delta_2} - 2\sqrt{\Delta_3} + 3)}{4} + \frac{\sqrt{\Delta_2\Delta_3}(2\sqrt{\Delta_2} + 1)(2\sqrt{\Delta_3} + 1)}{4} r \quad (5.14)$$

$$\begin{aligned} B_3 &= \frac{\sqrt{\Delta_2\Delta_3}(20\Delta_2\Delta_3 + 6\Delta_2\sqrt{\Delta_3} + 6\Delta_3\sqrt{\Delta_2} + 4\Delta_2 + 4\Delta_3 + 27\sqrt{\Delta_2\Delta_3} - 6\sqrt{\Delta_2} - 6\sqrt{\Delta_3} + 8)}{18} \\ &+ \frac{(\sqrt{\Delta_2} - 2)(\sqrt{\Delta_3} - 2)}{18} \frac{\sqrt{\Delta_2\Delta_3}(2\sqrt{\Delta_2} + 1)(2\sqrt{\Delta_3} + 1)}{4} r \end{aligned} \quad (5.15)$$

...

where

$$r \equiv \frac{3\eta^2 - 4\delta(\hat{\delta}_1 + \hat{\delta}_2 + \hat{\delta}_3 + \hat{\delta}_4) - 6(\hat{\delta}_1\hat{\delta}_3 + \hat{\delta}_2\hat{\delta}_4) + 12\hat{\delta}_1\hat{\delta}_4 + 3\hat{\delta}_2\hat{\delta}_3}{4\delta^2 - 9\eta^2} \quad (5.16)$$

and we introduced rescaled quantities: $\hat{\delta}_1 = \delta_1/(2\sqrt{\Delta_2} + 1)$, $\hat{\delta}_2 = \delta_2/\sqrt{\Delta_2}$, $\hat{\delta}_3 = \delta_3/\sqrt{\Delta_3}$, $\hat{\delta}_4 = \delta_4/(2\sqrt{\Delta_3} + 1)$. This is how (5.9) works in the $c = 1$ case.

Similarly, substituting $\Delta = \alpha_{3,1}^2 = \Delta_{3,1}(c = 1) = \Delta_{1,3}(c = 1) = 1$, one gets:

$$\begin{aligned} NB_3 &= -18(\Delta_1 - \Delta_2) \left(1 - 2(\Delta_1 + \Delta_2) + (\Delta_1 - \Delta_2)^2\right) (\Delta_3 - \Delta_4) \left(1 - 2(\Delta_3 + \Delta_4) + (\Delta_3 - \Delta_4)^2\right) \\ NB_4 &= 100(4 - \Delta_1 + \Delta_2)(4 + \Delta_3 - \Delta_4) \cdot NB_3, \\ &\dots \\ NB_k &\sim NB_3 \quad \text{for } k \geq 3 \end{aligned} \quad (5.17)$$

Thus, one observes that any zero of NB_3 remains zero of higher NB_k .

One can easily check that this remains true for other Kac zeroes.

5.4 Phenomenon 2: nested structure is not always enhanced at multiple (irregular) poles. $c = 7/10$ and $c = 1/2$ examples

At $c = 7/10$ there is a simple pole at $\Delta = 3/2$ on the third level, i.e. at B_3 , while in B_4 it becomes the double pole. This is because at $b = \sqrt{5}/2$ there is an additional degeneracy: $\Delta_{1,3} = \Delta_{4,1} = \frac{3}{2}$.

When all $\Delta_i = 3/2$, there is only a first-order zero in the numerators of both B_3 and B_4 , so that B_4 remains infinite, while B_3 is just ambiguous, which is not like the cases we considered earlier. For instance, in the Ashkin-Teller case a non-zero multiplicity of the pole immediately resulted into the non-zero multiplicity of the corresponding zero of the conformal block numerator. This means that the in $c = 7/10$ case the structure constants $C_{3/2,3/2}^{3/2}$ should vanish (while in the Ashkin-Teller case there is no need for this). In fact, this imposes restrictions on the rational conformal theories, since at rational values of the central charge there always emerge poles of higher multiplicities in the conformal block.

In $c = 7/10$ theory the problem emerges in the symmetric point, when all the external dimensions are equal to each other. However, a similar phenomenon takes place already in a simpler $c = 1/2$ theory though in a non-symmetric point. We now present this case in a little more detail.

At the central charge $c = 1/2$ the Kac determinants are

$$\begin{aligned} K_2 &= 2\Delta_0(16\Delta_0 - 1)(2\Delta_0 - 1), \\ K_3 &= 6\Delta_0(16\Delta_0 - 1)(2\Delta_0 - 1)^2(3\Delta_0 - 5), \\ K_4 &= 6\Delta_0(16\Delta_0 - 1)^2(2\Delta_0 - 1)^2(16\Delta_0 - 21)(3\Delta_0 - 5)(2\Delta_0 - 7), \\ &\dots \end{aligned} \quad (5.18)$$

The double zeroes in these formulas occur due to coincidence of the dimensions at $c = 1/2$: $\Delta_{1,3} = \Delta_{2,1} = \frac{1}{2}$ and $\Delta_{2,2} = \Delta_{1,2} = \frac{1}{16}$. However, these accidental enhanced zeroes do not produce extra poles in B_3 and B_4 : for arbitrary values of the four external dimensions the numerator of $B_3 \sim (2\Delta_0 - 1)$, and $B_4 \sim (16\Delta_0 - 1)(2\Delta_0 - 1)$, and this guarantees that the poles remain simple.

Nevertheless, the situation turns out to be not that simple. The remaining simple zero still needs to be compensated in the numerator, and now this is not universal. Indeed, let us consider the conformal block of $c = 1/2$ theory with first two external dimensions parameterized as $\Delta_1 = \frac{x^2-1}{48} + \frac{1}{3} + \frac{x}{6}$ and $\Delta_2 = \frac{x^2-1}{48}$, which guarantees that the zero is simple in the denominator of B_2 . However, in B_3 this does not provide an extra zero in addition to the factor $(2\Delta_0 - 1)$ which emerges independently of the external dimensions. This breaks the nested structure of (5.6), and one has to impose an additional restriction to avoid infinities.

5.5 Phenomenon 3: universality. $c = 1/2$ example

The leading behaviour is the vicinity of the singularity is

$$B_3 = \frac{(x+5)(x-1)(x+2)(\Delta_3 - \Delta_4)(2\Delta_3^2 - 4\Delta_3\Delta_4 + 2\Delta_4^2 - 3\Delta_3 - 3\Delta_4 + 1)}{3^4 2^3 (3\sigma - 7\delta)} \frac{1}{\epsilon} + O(\epsilon^0) \quad (5.19)$$

Hence, one can either choose a particular x , or specially match Δ_3 and Δ_4 .

In the first case, one can choose, for instance, $x = -2$. Then, the nested structure is restored: this condition is enough to cancel poles in B_3 , B_4 , However, the universality (5.9) is broken down similarly to the Ashkin-Teller case in the previous section: the conformal block looks like

$$\begin{aligned}
B_1 &= \frac{2\Delta_3 - \Delta_4 + 1}{4} \\
B_2 &= \frac{1}{7 \cdot 2^5} \left[36(\Delta_3^2 + \Delta_4^2 - \Delta_3\Delta_4) + 88\Delta_3 - 80\Delta_4 + 31 \right] + \frac{1}{7 \cdot 2^5} (12\Delta_3^2 - 24\Delta_3\Delta_4 + 12\Delta_4^2 - 8\Delta_3 - 8\Delta_4 + 1) r \\
B_{24 \geq k \geq 3} &= B_k^{(0)} + B_k^{(1)} r + B_k^{(2)} r_2 \\
&\dots
\end{aligned} \tag{5.20}$$

since the pole at $\Delta = 1/2$ becomes triple at level 25: $\Delta_{(5,5)} = 1/2$ at $c = 1/2$. Here

$$\begin{aligned}
r &\equiv \frac{\sigma - 28\delta_1 - 28\delta_2}{7\delta + 2\sigma} \\
r_2 &\equiv \frac{\delta_1 - \delta_2}{3\sigma - 7\delta}
\end{aligned} \tag{5.21}$$

In the second case, for matching Δ_3 and Δ_4 one can use the same parametrization: $\Delta_4 = \frac{y^2-1}{48} + \frac{1}{3} + \frac{y}{6}$ and $\Delta_3 = \frac{y^2-1}{48}$ so that

$$B_3 = \frac{5}{3^7 2^7} \frac{(x+5)(x-1)(x+2)(y+5)(y-1)(y+2)}{3\sigma - 7\delta} \frac{1}{\epsilon} + O(\epsilon^0) \tag{5.22}$$

and one suffices to choose $y = 1$, $y = -2$ or $y = -5$. However, in this case of two restricted dimensions and one matched to cancel the pole in the denominator, i.e. in the case of only one-parametric subspace in the 4-dimensional space $(\Delta_1, \Delta_2, \Delta_3, \Delta_4)$, one can observe a new phenomenon.

5.6 Phenomenon 4: additional universality of minimal models. Ising model example

In the case of the four external dimensions, parameterized by two variables (x, y) as above (so that appropriate zero occurs in the numerator), the conformal block is equal at, say, $x = -2$ to

$$\begin{aligned}
B_1 &= \frac{1-y}{2} \\
B_2 &= \frac{(y-1)(y-7)}{3 \cdot 2^6} \\
B_3 &= -\frac{5}{3^4 2^8} (y-1)(y-7)(y-13) - \frac{5}{3^4 2^6} (y-1)(y+2)(y+5)r \\
B_4 &= \frac{1}{7 \cdot 3^5 2^{13}} (y-1)(145y^3 - 4575y^2 + 58143y - 211825) + \frac{5}{3^5 2^8} (y-19)(y+2)(y+5)(y-1)r \\
&\dots
\end{aligned} \tag{5.23}$$

with

$$r = \frac{\delta_1 - \delta_2}{3\sigma - 7\delta} \tag{5.24}$$

One can see that with such a choice of the external dimensions, the coefficient B_2 is unambiguous, while the ambiguity goes away in higher B_k as soon as one chooses $y = 1, -2, -5$, since the coefficients in front of r are cancelled in such a case. Note now that the external dimensions (Δ_3, Δ_4) is $(0, 1/2)$ at $y = 1$, $(1/16, 1/16)$ at $y = -2$ and $(1/2, 0)$ at $y = -5$. These dimensions are exactly the ones of the Ising model: the model with the central charge $c = 1/2$, and the three primary operators: with dimensions 0, 1/2 and 1/16.

Thus, in the Ising model we encounter a new phenomenon: **the coefficient in front of r vanishes**, i.e.

$$\text{dependence on direction of approach to the Ising point disappears} \tag{5.25}$$

despite the Ising point lies on the ambiguity locus.

In fact, this remains the case for other minimal models (see an example of $c = 1$ case in Appendix B), and thus (if there are no other examples of this kind) can serve as still another definition of minimal models, entirely at the level of perturbative conformal blocks.

Let us see how it works with various 4-point conformal blocks in the vicinity of the Ising point.

$$\Delta_e^{(0)} = \frac{1}{16}, \Delta^{(0)} = \frac{1}{2}$$

$$\begin{aligned} B_2 &= \frac{\frac{9}{32}(2\sigma + 7\delta)\epsilon + O(\epsilon^2)}{2(2\sigma + 7\delta)\epsilon + O(\epsilon^2)} = \frac{9}{64} + O(\epsilon), \\ B_3 &= \frac{\frac{75}{128}(2\sigma + 7\delta)\left((3\sigma - 7\delta)\epsilon^2 + O(\epsilon^3)\right)}{6(2\sigma + 7\delta)(3\sigma - 7\delta)\epsilon^2 + O(\epsilon^3)} = \frac{25}{256} + O(\epsilon), \\ B_4 &= \frac{\frac{502047}{2048}(2\sigma + 7\delta)(3\sigma - 7\delta)\epsilon^2 + O(\epsilon^3)}{12 \cdot 273 \cdot (2\sigma + 7\delta)(3\sigma - 7\delta)\epsilon^2 + O(\epsilon^3)} = \frac{613}{8192} + O(\epsilon), \\ &\dots \end{aligned} \tag{5.26}$$

$$\Delta_e^{(0)} = \frac{1}{16}, \Delta^{(0)} = 0$$

$$\begin{aligned} B_2 &= \frac{\frac{1}{32}\delta\epsilon + O(\epsilon^2)}{2\delta\epsilon + O(\epsilon^2)} = \frac{1}{64} + O(\epsilon), \\ B_3 &= \frac{\frac{15}{32}\delta\epsilon + O(\epsilon^2)}{30\delta\epsilon + O(\epsilon^2)} = \frac{1}{64} + O(\epsilon), \\ B_4 &= \frac{-\frac{257985}{4096}\delta\epsilon + O(\epsilon^2)}{-4410\delta\epsilon + O(\epsilon^2)} = \frac{117}{8192} + O(\epsilon) \end{aligned} \tag{5.27}$$

again there is no r -dependence.

See more details about the Ising model in Appendix A.

5.7 Summary

In this section we originated a detailed examination of Kac zeroes, where most of conventionally studied conformal models are located, and where the standard near-divisor ambiguity arises, preventing definition of conformal blocks, both perturbative and non-perturbative, as the well-defined limit from non-singular expressions. We saw in the previous section 4 that this ambiguity can explain the apparent difference between available non-perturbative conformal blocks at the Ashkin-Teller point, and thus we believe that understanding of the near-divisor structure will be important for further development of non-perturbative CFT. As we explained, already at the first glance, this structure is quite interesting. Namely, we described four non-trivial phenomena specific for conformal blocks and emphasizing that they are far from exhibiting a *generic* behavior near the singularity: quite the opposite, their behavior is adjusted in a very special way, which should be better studied and interpreted.

So, when we look at the Kac zero at level l , the corresponding singularity appears first in the coefficient B_l , and to avoid singularity one should adjust the external dimensions Δ_i to make the numerator of NB_l vanishing as well. Then the pole continues to be present in all the higher coefficients B_k with $k \geq l$. But:

- **Phenomenon 1.** There is a nested structure in conformal blocks: the zero of the *numerator* NB_l at this pole is also present in all higher numerators NB_k with $k \geq l$. This means that when the Kac zero is simple, it is enough to adjust external dimensions only once, in the first relevant B_l , and then the entire conformal block is non-singular: the zero of the first relevant numerator at the simple Kac zero is inherited by all other numerators.

However, for many interesting choices of internal (intermediate) dimension Δ , the zero of Kac determinant is *not* simple, there are "accidental" coincidences of different zeroes in the numerators of B_k with $k \geq l$ for all conformal theories with rational central charges. Then

- **Phenomenon 2.** The nested structure gets broken, in the sense that the zero in the numerator of conformal block remains simple. Thus, for accidentally degenerate Kac zeroes we encounter "naked singularities" like in *generic* function on the moduli space.

However, there is a notable exception from this pessimistic picture:

- **Phenomenon 3.** The nested structure is typically restored, as soon as one additionally adjusts the cancellation of the multiple pole in the first relevant B_l . In this case, the intermediate Δ is such that the Kac zeroes are not simple, but one and the same choice of external Δ_i provides all the numerators with the zeroes of exactly the right order to eliminate the singularity. Then, the ambiguity in the conformal block is piece-wise universal: if the simple Kac zero emerges at some l_1 , it becomes a double zero at some l_2 etc (as soon as there emerges an "accidental" double pole, sooner or later there emerge all higher multiplicities), all $B_{k < l_1}$ are unambiguous, $B_{l_1 \leq k < l_2}$ are universal linear functions of one parameter describing approach to the singularity locus, $B_{l_2 \leq k < l_3}$ are universal functions which are linear combinations of two parameters etc.

More than that:

- **Phenomenon 4.** At the minimal model points the near-divisor ambiguity *disappears*: the limit does *not* depend on the direction on the moduli space, from which we approach the minimal model points. This is probably the most spectacular manifestation of how special the minimal models really are from the point of view of conformal block properties, and relation of this property to many others (like the finiteness of block quantity, needed for the conformal bootstrap at these points) still remains to be understood.

It is not clear if these four phenomena provide an exhaustive description of peculiarities of the conformal block behavior at the Kac divisor, even in the simplest 4-point spherical case. Very interesting should be extension of this study to more points and higher genera. And, of course, the crucial question is the implication for non-perturbative corrections. All this remains to be thoroughly investigated.

6 Null-vectors, equations and hidden parameters

In the previous sections we studied ambiguities that appear at the singularity locus of the conformal block and clarified their origin and peculiar properties. Here we consider the specific conformal blocks when some of the external dimensions correspond to a degenerate vector, since in this case one can deal with the conformal block not as a series but a space of solutions to a differential equation. Hence, one can check if it is possible to find some non-perturbative hidden parameters, i.e. if a conformal block can be presented as a linear combination of solutions with some arbitrary coefficients.

We start the simplest example of the vector degenerate at the second level. This vector is of the form $\tilde{V} = (\xi L_{-1}^2 - L_{-2})V_\Delta$ and there are two non-trivial conditions: $L_1 \tilde{V} = 0$ and $L_2 \tilde{V} = 0$. They imply respectively that

$$\xi = \frac{3}{2(2\Delta + 1)} \quad (6.1)$$

and

$$8\Delta + c = 12\xi\Delta \quad (6.2)$$

or, together

$$\Delta = \frac{5 - c \pm \sqrt{(c-1)(c-25)}}{16} \quad (6.3)$$

Parameterizing the central charge and dimension as in (3.26), we obtain four solutions:

$$\boxed{\begin{cases} \alpha = \frac{1}{2b} \\ \xi = b^2 \end{cases}}, \quad \begin{cases} \alpha = -\frac{b}{2} \\ \xi = \frac{1}{b^2} \end{cases}, \quad \begin{cases} \alpha = \frac{3b}{2} - \frac{1}{b} \\ \xi = b^2 \end{cases}, \quad \begin{cases} \alpha = b - \frac{3}{2b} \\ \xi = \frac{1}{b^2} \end{cases} \quad (6.4)$$

In what follows we work with the first of these four solutions (boxed), so that the original highest weight primary $V_{1/2b}$ of degenerate Verma module has dimension

$$\Delta_{1/2b} = -\frac{1}{2} + \frac{3}{4b^2} \quad (6.5)$$

The conformal Ward identities imply that 4-point correlators $\Psi_4(x, \bar{x})$ with insertion of this degenerate primary at point x satisfy peculiar differential equations, see [1]:

$$\left\{ b^2 x(x-1) \partial_x^2 + (2x-1) \partial_x + \Delta_{1/2b} + \frac{\Delta_1}{x} - \frac{\Delta_3}{x-1} - \Delta_4 \right\} \Psi_4(x) = 0 \quad (6.6)$$

where we suppressed the dependence on \bar{x} . In the free field realization of conformal field theory this constraint is imposed almost automatically, see [61], and this is also easily seen from the Dotsenko-Fateev β -ensemble representation of the corresponding conformal blocks, [61].

Conjugation with a factor $x^\alpha(1-x)^\beta$ with specially adjusted α and β converts (6.6) into an ordinary hypergeometric equation with the solution

$$\begin{aligned} \Psi_4(x) &= x^{\alpha_1/b} (1-x)^{\alpha_3/b} F(A, B; C; x) \\ A &= \frac{1}{2b^2} + \frac{\alpha_1}{b} + \frac{\alpha_2}{b} - \frac{\alpha_3}{b} \\ B &= \frac{1}{b} \sum_{i=1}^3 \alpha_i + 2\Delta_{1/2b}, \quad C = \frac{1}{b^2} + \frac{2\alpha_1}{b} \end{aligned} \quad (6.7)$$

Equations (6.6), (6.7) are consistent with generic formulas (3.22)-(3.23) only if the dimensions Δ_1 and Δ are related (in parametrization (3.26)) by the fusion rule

$$\alpha = \alpha_1 \pm \frac{1}{2b} \quad (6.8)$$

where two choices of the sign correspond to the two linearly independent solutions of (6.6) and in the case of the sign “minus” in (6.8) one has to choose in (6.7) instead of $F(A, B; C; x)$ the other solution to the hypergeometric equation so that $\alpha_1 \rightarrow b - 1/b - \alpha_1$ in $\Psi_4(x)$ in (6.7).

One can easily check directly that the conformal block from the r.h.s. of (6.10)

$$\mathcal{B}_{\Delta_\alpha}^{(1,1/2b;34)}(x) = x^{\Delta_\alpha - \Delta_1 - \Delta_{1/2b}} \left(1 + \frac{(\Delta_\alpha + \Delta_{1/2b} - \Delta_1)(\Delta_\alpha + \Delta_3 - \Delta_4)}{2\Delta_\alpha} x + \dots \right) \stackrel{(6.8)}{=} \Psi_4(x) \quad (6.9)$$

which solves (6.6). Formula (3.1) now acquires the form

$$\begin{aligned} \langle V_1(0) V_{1/2b}(x) V_3(1) V_4(\infty) \rangle &= \sum_{\Delta, \bar{\Delta}} C_{1,1/2b}^{\Delta, \bar{\Delta}} C_{34}^{\Delta, \bar{\Delta}} \mathcal{B}_\Delta^{(1,1/2b;34)}(x) \bar{\mathcal{B}}_{\bar{\Delta}}^{(1,1/2b;34)}(\bar{x}) = \\ &= \sum_{\substack{\alpha = \alpha_1 \pm \frac{1}{2b} \\ \bar{\alpha} = \bar{\alpha}_1 \pm \frac{1}{2b}}} C_{1,1/2b}^{\Delta_\alpha, \bar{\Delta}_{\bar{\alpha}}} C_{34}^{\Delta_\alpha, \bar{\Delta}_{\bar{\alpha}}} \mathcal{B}_{\Delta_\alpha}^{(1,1/2b;34)}(x) \bar{\mathcal{B}}_{\bar{\Delta}_{\bar{\alpha}}}^{(1,1/2b;34)}(\bar{x}) \end{aligned} \quad (6.10)$$

since **only for the choice (6.8) the structure constant $C_{1,1/2b}^{\Delta_\alpha}$ is non-vanishing** [1]. Here we obtained this fact indirectly by solving the equation for the correlator. One can derive this fact straightforwardly using the β -ensemble representation for the conformal blocks [61].

Thus, two solutions of the equation for the degenerate conformal block describes two *different* conformal blocks, the only two with non-zero structure constants. One can easily see also (6.8) from their asymptotics: since the conformal block behaves at small x like $x^{\Delta - \Delta_1 - \Delta_2}$, one gets (6.8) from the asymptotic behaviours $\Psi_4^{(1)}(x) \sim x^{\alpha_1/b}$ and $\Psi_4^{(2)}(x) \sim x^{1-1/b^2-\alpha_1/b}$.

This system of two conformal blocks is self-consistent: it survives the modular transformation (3.3): $x \rightarrow$

$1 - x$, $\Delta_1 \leftrightarrow \Delta_3$. Indeed (see [60, (9.131(1,2))]),

$$\begin{aligned}
\Psi_4^{(1)}(\Delta_3, \Delta_2, \Delta_1, \Delta_4; 1 - x) &= \frac{\Gamma\left(\frac{2\alpha_3}{b} + \frac{1}{b^2}\right)\Gamma\left(\frac{1}{b^2} + \frac{2\alpha_1}{b} - 1\right)}{\Gamma\left(\frac{3}{2b^2} - 1 + \frac{\alpha_1}{b} + \frac{\alpha_3}{b} + \frac{\alpha_4}{b}\right)\Gamma\left(\frac{1}{2b^2} + \frac{\alpha_1}{b} + \frac{\alpha_3}{b} - \frac{\alpha_4}{b}\right)} \Psi_4^{(1)}(\Delta_1, \Delta_2, \Delta_3, \Delta_4; x) + \\
&+ \frac{\Gamma\left(\frac{2\alpha_3}{b} + \frac{1}{b^2}\right)\Gamma\left(1 - \frac{1}{b^2} - \frac{2\alpha_1}{b}\right)}{\Gamma\left(1 - \frac{1}{2b^2} + \frac{\alpha_3}{b} - \frac{\alpha_1}{b} - \frac{\alpha_4}{b}\right)\Gamma\left(\frac{1}{2b^2} + \frac{\alpha_3}{b} + \frac{\alpha_4}{b} - \frac{\alpha_1}{b}\right)} \Psi_4^{(2)}(\Delta_1, \Delta_2, \Delta_3, \Delta_4; x) \\
\Psi_4^{(2)}(\Delta_3, \Delta_2, \Delta_1, \Delta_4; 1 - x) &= \frac{\Gamma\left(2 - \frac{2\alpha_3}{b} - \frac{1}{b^2}\right)\Gamma\left(\frac{1}{b^2} - 1 + \frac{2\alpha_1}{b}\right)}{\Gamma\left(\frac{1}{2b^2} + \frac{\alpha_1}{b} + \frac{\alpha_4}{b} - \frac{\alpha_3}{b}\right)\Gamma\left(1 - \frac{1}{2b^2} + \frac{\alpha_1}{b} - \frac{\alpha_3}{b} - \frac{\alpha_4}{b}\right)} \Psi_4^{(1)}(\Delta_1, \Delta_2, \Delta_3, \Delta_4; x) + \\
&+ \frac{\Gamma\left(2 - \frac{2\alpha_3}{b} - \frac{1}{b^2}\right)\Gamma\left(1 - \frac{1}{b^2} - \frac{2\alpha_1}{b}\right)}{\Gamma\left(2 - \frac{3}{2b^2} - \frac{\alpha_1}{b} - \frac{\alpha_3}{b} - \frac{\alpha_4}{b}\right)\Gamma\left(1 - \frac{1}{2b^2} + \frac{\alpha_1}{b} + \frac{\alpha_3}{b} - \frac{\alpha_4}{b}\right)} \Psi_4^{(2)}(\Delta_1, \Delta_2, \Delta_3, \Delta_4; x)
\end{aligned} \tag{6.11}$$

i.e. the modular kernel in this case is the 2×2 matrix

$$\begin{pmatrix}
\frac{\Gamma\left(\frac{2\alpha_3}{b} + \frac{1}{b^2}\right)\Gamma\left(\frac{1}{b^2} + \frac{2\alpha_1}{b} - 1\right)}{\Gamma\left(\frac{3}{2b^2} - 1 + \frac{\alpha_1}{b} + \frac{\alpha_3}{b} + \frac{\alpha_4}{b}\right)\Gamma\left(\frac{1}{2b^2} + \frac{\alpha_1}{b} + \frac{\alpha_3}{b} - \frac{\alpha_4}{b}\right)} & \frac{\Gamma\left(\frac{2\alpha_3}{b} + \frac{1}{b^2}\right)\Gamma\left(1 - \frac{1}{b^2} - \frac{2\alpha_1}{b}\right)}{\Gamma\left(1 - \frac{1}{2b^2} + \frac{\alpha_3}{b} - \frac{\alpha_1}{b} - \frac{\alpha_4}{b}\right)\Gamma\left(\frac{1}{2b^2} + \frac{\alpha_3}{b} + \frac{\alpha_4}{b} - \frac{\alpha_1}{b}\right)} \\
\frac{\Gamma\left(2 - \frac{2\alpha_3}{b} - \frac{1}{b^2}\right)\Gamma\left(\frac{1}{b^2} - 1 + \frac{2\alpha_1}{b}\right)}{\Gamma\left(\frac{1}{2b^2} + \frac{\alpha_1}{b} + \frac{\alpha_4}{b} - \frac{\alpha_3}{b}\right)\Gamma\left(1 - \frac{1}{2b^2} + \frac{\alpha_1}{b} - \frac{\alpha_3}{b} - \frac{\alpha_4}{b}\right)} & \frac{\Gamma\left(2 - \frac{2\alpha_3}{b} - \frac{1}{b^2}\right)\Gamma\left(1 - \frac{1}{b^2} - \frac{2\alpha_1}{b}\right)}{\Gamma\left(2 - \frac{3}{2b^2} - \frac{\alpha_1}{b} - \frac{\alpha_3}{b} - \frac{\alpha_4}{b}\right)\Gamma\left(1 - \frac{1}{2b^2} + \frac{\alpha_1}{b} + \frac{\alpha_3}{b} - \frac{\alpha_4}{b}\right)}
\end{pmatrix} \tag{6.12}$$

We demonstrate how all this works in the minimal models in examples of $c = 1/2$ (the Ising model) and $c = 1$ in Appendices A and B respectively.

Similarly one can deal with conformal blocks degenerate at higher levels. In these case the order of the corresponding differential equation is higher, but again is equal exactly to the number of conformal blocks with non-zero structure constants. The modular matrix also accordingly increases its size, see examples in [1]. Hence, no hidden parameters are emerge in this way. In fact, this is not surprising, since the differential equations are w.r.t. the variable x , and, as we already stressed, the x -behaviour of the conformal block is not expected to depend on non-perturbative (hidden) parameters.

7 Conclusion

The main goal of this paper is to urge the study of the non-perturbative conformal block as a function of *all* its variables: coordinates, external and internal dimensions and the central charge. The first question to ask here is if there are some *extra* parameters, besides already enumerated, on which the non-perturbative quantity usually depends, which are not seen at the perturbative level like the theta-angle in instanton calculus. In conformal block story, the main ambiguity is the overall normalization, which can be a *function* of dimensions and central charge (thus in fact can contain infinitely many extra parameters). The lack of control over such normalization factors is the main current problem in relating different efficient non-perturbative approaches, say, to constructing the modular kernel: the $SL_q(2)$ method of Ponsot-Teschner [41], the matrix model approach of [57] uncovering the Stokes (wall crossing) phenomena and relating cluster variables to check-exponents of [62], and the Painleve equation method of [59]. In the present paper we did not attack this problem of Δ -dependence directly: instead we tried to look for extra non-perturbative parameters, considering the x -dependence of the simplest (4-point spherical) conformal block. We looked mostly at two obvious places: at discrepancy between the explicitly known non-perturbative Zamolodchikov and elliptic-integral answers at the Ashkin-Teller point (they are different functions of x), and at the higher order differential equation in x , which conformal block satisfies when one of the vertex operators is degenerate (e.g. in the Ising and other minimal models), both cases could seem to imply the existence of extra parameters. As we explained, this is, however, not the case. The discrepancy at the Ashkin-Teller point (and in many similar cases) is in fact just the ordinary ambiguity at the singularity divisor for a function of many variables (dimensions and central charge), and no extra parameters are present. The case with many, rather than one, solutions to a higher order differential equation is resolved *not* by introduction of extra variables, but by the fact that $B(x)$ actually does *not* satisfy such equations when just one

external dimension is fixed: in fact, the equation is true for the conformal block only when the internal dimension Δ is fixed as well [61] (this is a very important feature of the conformal block, which is often overlooked or underestimated). In result, the extra solutions are in fact describing not an ambiguity in the function $B_\Delta(x)$, but the *other* conformal blocks $B_{\Delta'}(x)$ with the *other* allowed values of internal dimension Δ' : there exactly as many of them as the degeneration level of vertex operator and the order of the differential equation.

Thus we found *no* evidence for extra non-perturbative parameters in the x -dependence of conformal block. This could seem obvious from the very beginning: as we already mentioned, there is a belief that the 4-point spherical conformal block is actually a Belyi function (i.e. has only non-essential singularities at three points $x = 0, 1, \infty$, and ramification orders are integer for rational conformal models). This, in turn, can be attributed either to the fact that $2d$ CFT is actually a free field theory [1] so that there is actually no interaction and no reason for *real* non-perturbative effects to exist, or to another fact: that it is conformal, and then no non-trivial dependence is allowed for a function of the single dimensional parameter x .

We believe that our simple consideration sheds some new light on the problem of non-perturbative conformal blocks and can help to attract new attention to this extremely interesting problem. Conformal blocks are the crucially important special function of the string era, and they should be thoroughly investigated and understood.

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Appendix A. Ising model

The critical behaviour of the Ising model is described by the central charge $c = 1/2$, and we choose $b = \sqrt{3}/2$. There three primary fields with the dimensions: $\Delta_I = 0$ (i.e. $\alpha_I = 0$ or $-\sqrt{3}/6$), $\Delta_\psi = 1/2$ (i.e. $\alpha_\psi = \sqrt{3}/3$ or $-\sqrt{3}/2$), $\Delta_\sigma = 1/16$ (i.e. $\alpha_\sigma = \sqrt{3}/12$ or $-\sqrt{3}/4$). The first field is degenerate at the first level, the second and the third ones are degenerate at the second level. One can calculate the conformal blocks in different cases. For instance, consider $\mathcal{B}(\Delta_1, \Delta_2, \Delta_3, \Delta_4; \Delta; c; x)$ and suppose that the intermediate dimension $\Delta = \Delta_\psi = 1/2$. It leads to poles at all levels higher than one. The condition of canceling this pole by matching the dimensions Δ_1 and Δ_2 requires them be either $1/2$ and 0 or $\Delta_1 = \Delta_2 = 1/16$ (if one restricts himself with the spectrum of three fields above). Hence, the correlators of fields $\langle \psi\psi I \rangle$ and $\langle \psi\sigma\sigma \rangle$ are non-zero. Similarly considering the conformal block with the intermediate dimension $\Delta = \Delta_\sigma = 1/16$, one finds the non-zero correlator $\langle \psi\psi I \rangle$. This fixes the operator product expansion (OPE) of fields.

Now consider the conformal block with the field $\psi(x)$ at point x (remind that, in our notation, this corresponds to Δ_2 , while Δ_1 , Δ_3 and Δ_4 corresponds to the fields at points 0 , 1 and ∞ respectively). This field is degenerate at the second level and the conformal block satisfies a second order differential equation provided $\alpha = \alpha_1 \pm 1/2b$ (see s.6). This condition can be satisfied only for the pairs (α, α_1) either $(\alpha_\sigma, \alpha_\sigma)$ or (α_ψ, α_I) which again fixes the OPE. Consider the correlator $\langle \sigma(0)\psi(x)\sigma(1)\psi(\infty) \rangle$. It is described by the values $\alpha_1 = -\sqrt{3}/4$, $\alpha_2 = \sqrt{3}/3$, $\alpha_3 = -\sqrt{3}/4$, $\alpha_4 = -\sqrt{3}/2$. The corresponding conformal block $\mathcal{B}(x)$ satisfies the differential equation (6.6)

$$\left[\frac{3}{4}x(1-x)\partial_x^2 + (2x-1)\partial_x + \frac{1}{16}\left(\frac{1}{x} + \frac{1}{1-x}\right) \right] \mathcal{B}(x) = 0 \quad (\text{A1})$$

This equation is hypergeometric and has two solutions:

$$\begin{aligned} \mathcal{B}^{(1)}(x) &= \frac{1-2x}{\sqrt{x(1-x)}} \\ \mathcal{B}^{(2)}(x) &= [x(1-x)]^{1/6} F(1/3, 2; 5/3; x) \end{aligned} \quad (\text{A2})$$

where $F(a, b; c; x)$ is the hypergeometric function. The first solution corresponds to the behaviour at small x

$$\mathcal{B}^{(1)}(x) = \frac{1}{\sqrt{x}} \left(1 - \frac{3}{2}x - \frac{5}{8}x^2 - \frac{7}{16}x^3 - \frac{45}{128}x^4 + \dots \right) \quad (\text{A3})$$

of the conformal block. The multiplier $1/\sqrt{x}$ comes from the usual pre-factor $x^{\Delta-\Delta_1-\Delta_2}$ of the conformal block and implies (which we already established from the OPE earlier) that $\Delta = \Delta_\sigma = 1/16$. The second solution ought to describe an intermediate field with dimension $\Delta = 35/48$. This field is absent in the spectrum which means that the corresponding structure constant is zero.

The expansion (A3) of the conformal block should be compared with its generic expansion (3.21). One can check that they coincide independently on the way the singularity is resolved, i.e. if one considers a vicinity of the point $\Delta_1 = 1/16$, $\Delta_2 = 1/2$, $\Delta_3 = 16$, $\Delta_4 = 1/2$, $\Delta = 1/16$, the leading order in ϵ does not depend on the way of approaching the singularity at all (see s.5.6):

$$(3.21) = 1 - \frac{3}{2}x - \frac{5}{8}x^2 - \frac{7}{16}x^3 - \frac{45}{128}x^4 + \dots \quad (\text{A4})$$

Note that $\mathcal{B}^{(1)}(x)$ is consistently invariant with respect to the duality transformation $x \rightarrow 1-x$. Similarly invariant is the differential equation (A1), though the second solution is not: it transforms through itself and the first solution [60, 9.131(2)]:

$$F(1/3, 2; 5/3; x) = F(1/3, 2; 5/3; 1-x) + \frac{\Gamma(2/3)^3}{\sqrt{3}\pi} \frac{1}{[x(1-x)]^{1/6}} \mathcal{B}^{(1)}(x) \quad (\text{A5})$$

This means that the duality matrix is triangle.

Now consider another possible 4-point correlator in this theory: $\langle \sigma\sigma\sigma\sigma \rangle$. One can similarly write down the differential equation

$$\left[\frac{4}{3}x(1-x)\partial_x^2 + (2x-1)\partial_x + \frac{1}{16} \left(\frac{1}{x} + \frac{1}{1-x} \right) \right] \mathcal{B}(x) = 0 \quad (\text{A6})$$

This equation has two solutions:

$$\begin{aligned} \mathcal{B}^{(1)}(x) &= \frac{\sqrt{\sqrt{x}+1} + \sqrt{\sqrt{x}-1}}{[x(1-x)]^{1/8}} \\ \mathcal{B}^{(2)}(x) &= \frac{\sqrt{\sqrt{x}+1} - \omega\sqrt{\sqrt{x}-1}}{[x(1-x)]^{1/8}} \end{aligned} \quad (\text{A7})$$

where $\omega \equiv \exp(\pm\pi i/2)$ is a square root of -1 (plus or minus depends on the chosen branch of \sqrt{x}). The first solution has the small- x expansion

$$\mathcal{B}^{(1)}(x) \sim \frac{1}{x^{1/8}} \left(1 + \frac{1}{64}x^2 + \frac{1}{64}x^3 + \frac{117}{8192}x^4 + \frac{53}{4096}x^5 + \dots \right) \quad (\text{A8})$$

and corresponds to the intermediate field I with dimension $\Delta = \Delta_I = 0$. The second solution has the small- x expansion

$$\mathcal{B}^{(2)}(x) \sim \frac{\sqrt{x}}{x^{1/8}} \left(1 + \frac{1}{4}x + \frac{9}{64}x^2 + \frac{25}{256}x^3 + \frac{613}{8192}x^4 \right) \quad (\text{A9})$$

These two expansions as before reproduce the correct result (A3) independently on the way of resolving the singularity.

Since, in this case, the both solutions correspond to the "physical" conformal blocks, this is not surprising that the duality transformation acts as a matrix on these two solutions:

$$\begin{aligned} \mathcal{B}^{(1)}(1-x) &= \frac{1}{\sqrt{2}} \left(\mathcal{B}^{(1)}(x) + \frac{\omega}{2} \mathcal{B}^{(2)}(x) \right) \\ \mathcal{B}^{(2)}(1-x) &= \sqrt{2} \left(\mathcal{B}^{(1)}(x) - i\frac{\omega}{2} \mathcal{B}^{(2)}(x) \right) \end{aligned} \quad (\text{A10})$$

i.e. the duality matrix is

$$S = \sqrt{2} \begin{pmatrix} 1 & \frac{\omega}{2} \\ 2 & -i\omega \end{pmatrix} \quad (\text{A11})$$

The last non-trivial correlator in the Ising model is $\langle \psi\psi\psi\psi \rangle$. It is described by the differential equation

$$\left[\frac{3}{4}x(1-x)\partial_x^2 + (2x-1)\partial_x + \frac{1}{2} \left(\frac{1}{x} + \frac{1}{1-x} \right) \right] \mathcal{B}(x) = 0 \quad (\text{A12})$$

Again, only one of the two solutions of this equation is relevant to the Ising model, it is

$$\mathcal{B}(x) = \frac{1 - x + x^2}{x(1 - x)} \quad (\text{A13})$$

This solution is modular invariant and its small- x expansion gives the conformal block (A3) independently on the way of resolving the singularity:

$$\mathcal{B}(x) \sim \frac{1}{x} (1 + x^2 + x^3 + x^4 + \dots) \quad (\text{A14})$$

where the common factor $1/x$ implies that this conformal block describes the intermediate field of dimension $\Delta = \Delta_I = 0$ as it should be.

Appendix B. $c = 1$, $\Delta_e = \frac{1}{4}$

Let us consider the minimal model with $c = 1$, it can be obtained from the series of minimal models $(m, m+1)$ in the limit $m \rightarrow \infty$ [46, App.B], and one easily construct the conformal block of four fields with $\Delta_{(1,2)} = 1/4$, since they are degenerate at the second level. Solving equation (6.6) with $b = 1$ gives two solutions which correspond to the conformal block with internal dimensions $\Delta = 0$:

$$\mathcal{B}_{\Delta=0} \left(\Delta_i = \frac{1}{4}, c = 1 \mid x \right) = \frac{1}{\sqrt{x}} \left(1 + \frac{x^2}{8} + \frac{x^3}{8} + \frac{15x^4}{128} + \dots \right) = \frac{1}{2\sqrt{x}} \left(\sqrt{1-x} + \frac{1}{\sqrt{1-x}} \right) \quad (\text{B1})$$

and $\Delta = 1$

$$\mathcal{B}_{\Delta=1} \left(\Delta_i = \frac{1}{4}, c = 1 \mid x \right) = \sqrt{x} \left(1 + \frac{x}{2} + \frac{3x^2}{8} + \frac{5x^3}{16} + \frac{35x^4}{128} + \dots \right) = \frac{\sqrt{x}}{\sqrt{1-x}} \quad (\text{B2})$$

The modular transformation $x \rightarrow 1 - x$ acts on the doublet

$$\begin{pmatrix} \frac{1}{2} \sqrt{\frac{1-x}{x}} + \frac{1}{2} \sqrt{\frac{1}{x(1-x)}} \\ \sqrt{\frac{x}{1-x}} \end{pmatrix} \quad (\text{B3})$$

by the matrix

$$\begin{pmatrix} 1/2 & 3/4 \\ 1 & -1/2 \end{pmatrix} \quad (\text{B4})$$

From bilinear combination of these one can construct a modular invariant by adjusting the coefficient:

$$\frac{1}{|x|} |\mathcal{B}_0|^2 + \frac{3}{4} |x| \cdot |\mathcal{B}_1|^2 = \frac{|1 - \frac{x}{2}|^2 + \frac{3}{4} \cdot |x|^2}{|x| \cdot |1 - x|} \sim \frac{|x|}{|1 - x|} + \frac{|1 - x|}{|x|} + \frac{1}{|x(1 - x)|} \quad (\text{B5})$$

This answer coincides with [46].

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